Definition of the Limit of Function:

We now state the precise definition of the limit of a function f at a point c. It is important to note that in this definition, it is immaterial whether f is defined at c or not. In any case, we exclude c from consideration in the determination of the limit.

Definition Let $A \subseteq \mathbb{R}$, and let c be a cluster point of A. For a function $f : A \to \mathbb{R}$, a real number L is said to be a **limit of f at c** if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

Remarks **(a)** Since the value of δ usually depends on ε, we will sometimes write $\delta(\varepsilon)$ instead of δ to emphasize this dependence.
(b) The inequality $0 < |x - c|$ is equivalent to saying $x \neq c$.

If L is a limit of f at c, then we also say that f **converges to L at c**. We often write

$$L = \lim_{x \to c} f(x) \quad \text{or} \quad L = \lim_{x \to c} f.$$

We also say that "$f(x)$ approaches L as x approaches c." (But it should be noted that the points do not actually move anywhere.) The symbolism

$$f(x) \to L \quad \text{as} \quad x \to c$$

is also used sometimes to express the fact that f has limit L at c.

If the limit of f at c does not exist, we say that f **diverges** at c.

Our first result is that the value L of the limit is uniquely determined. This uniqueness is not part of the definition of limit, but must be deduced.

Theorem *If $f : A \to \mathbb{R}$ and if c is a cluster point of A, then f can have only one limit at c.*

Proof. Suppose that numbers L and L' satisfy Definition 4.1.4. For any $\varepsilon > 0$, there exists $\delta(\varepsilon/2) > 0$ such that if $x \in A$ and $0 < |x - c| < \delta(\varepsilon/2)$, then $|f(x) - L| < \varepsilon/2$. Also there exists $\delta'(\varepsilon/2)$ such that if $x \in A$ and $0 < |x - c| < \delta'(\varepsilon/2)$, then $|f(x) - L'| < \varepsilon/2$. Now let $\delta := \inf\{\delta(\varepsilon/2), \delta'(\varepsilon/2)\}$. Then if $x \in A$ and $0 < |x - c| < \delta$, the Triangle Inequality implies that

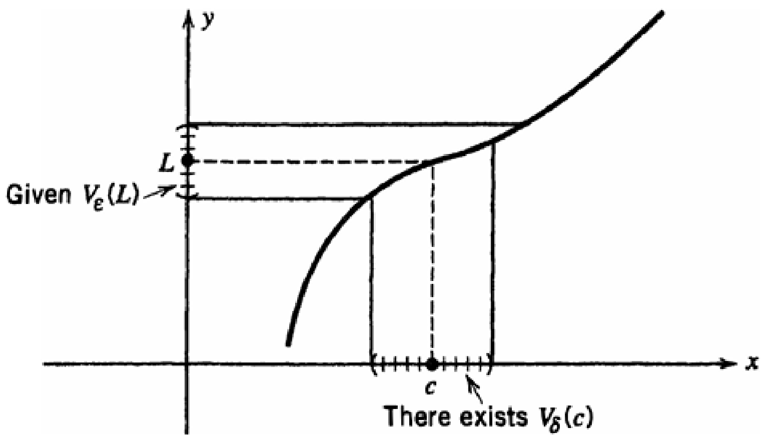
$$|L - L'| \leq |L - f(x)| + |f(x) - L'| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $L - L' = 0$, so that $L = L'$.

The definition of limit can be very nicely described in terms of neighborhoods. We observe that because

$$V_\delta(c) = (c - \delta, c + \delta) = \{x : |x - c| < \delta\},$$

the inequality $0 < |x - c| < \delta$ is equivalent to saying that $x \neq c$ and x belongs to the δ-neighborhood $V_\delta(c)$ of c. Similarly, the inequality $|f(x) - L| < \varepsilon$ is equivalent to saying that $f(x)$ belongs to the ε-neighborhood $V_\varepsilon(L)$ of L. In this way, we obtain the following result. The reader should write out a detailed argument to establish the theorem.



The limit of f at c is L

Theorem *Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. Then the following statements are equivalent.*

(i) $\lim_{x \to c} f(x) = L.$

(ii) *Given any ε-neighborhood $V_\varepsilon(L)$ of L, there exists a δ-neighborhood $V_\delta(c)$ of c such that if $x \neq c$ is any point in $V_\delta(c) \cap A$, then $f(x)$ belongs to $V_\varepsilon(L)$.*

Examples **(a)** $\lim_{x \to c} b = b.$

To be more explicit, let $f(x) := b$ for all $x \in \mathbb{R}$. We want to show that $\lim_{x \to c} f(x) = b$. If $\varepsilon > 0$ is given, we let $\delta := 1$. (In fact, any strictly positive δ will serve the purpose.) Then if

$0 < |x - c| < 1$, we have $|f(x) - b| = |b - b| = 0 < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude from Definition 4.1.4 that $\lim_{x \to c} f(x) = b$.

(b) $\lim_{x \to c} x = c.$

Let $g(x) := x$ for all $x \in \mathbb{R}$. If $\varepsilon > 0$, we choose $\delta(\varepsilon) := \varepsilon$. Then if $0 < |x - c| < \delta(\varepsilon)$, we have $|g(x) - c| = |x - c| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we deduce that $\lim_{x \to c} g = c$.

(c) $\lim_{x \to c} x^2 = c^2.$

Let $h(x) := x^2$ for all $x \in \mathbb{R}$. We want to make the difference

$$\left|h(x) - c^2\right| = \left|x^2 - c^2\right|$$

less than a preassigned $\varepsilon > 0$ by taking x sufficiently close to c. To do so, we note that $x^2 - c^2 = (x + c)(x - c)$. Moreover, if $|x - c| < 1$, then

$$|x| < |c| + 1 \qquad \text{so that} \qquad |x + c| \le |x| + |c| < 2\,|c| + 1.$$

Therefore, if $|x - c| < 1$, we have

$$\left|x^2 - c^2\right| = |x + c||x - c| < (2\,|c| + 1)\,|x - c|. \tag{1}$$

Moreover this last term will be less than ε provided we take $|x - c| < \varepsilon/(2|c| + 1)$. Consequently, if we choose

$$\delta(\varepsilon) := \inf\left\{1, \frac{\varepsilon}{2|c| + 1}\right\},$$

then if $0 < |x - c| < \delta(\varepsilon)$, it will follow first that $|x - c| < 1$ so that (1) is valid, and therefore, since $|x - c| < \varepsilon/(2|c| + 1)$ that

$$|x^2 - c^2| < (2|c| + 1)|x - c| < \varepsilon.$$

Since we have a way of choosing $\delta(\varepsilon) > 0$ for an arbitrary choice of $\varepsilon > 0$, we infer that $\lim_{x\to c} h(x) = \lim_{x\to c} x^2 = c^2$.

(d) $\lim_{x\to c} \frac{1}{x} = \frac{1}{c}$ if $c > 0$.

Let $\varphi(x) := 1/x$ for $x > 0$ and let $c > 0$. To show that $\lim_{x\to c} \varphi = 1/c$ we wish to make the difference

$$\left|\varphi(x) - \frac{1}{c}\right| = \left|\frac{1}{x} - \frac{1}{c}\right|$$

less than a preassigned $\varepsilon > 0$ by taking x sufficiently close to $c > 0$. We first note that

$$\left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{1}{cx}(c - x)\right| = \frac{1}{cx}|x - c|$$

for $x > 0$. It is useful to get an upper bound for the term $1/(cx)$ that holds in some neighborhood of c. In particular, if $|x - c| < \frac{1}{2}c$, then $\frac{1}{2}c < x < \frac{3}{2}c$ (why?), so that

$$0 < \frac{1}{cx} < \frac{2}{c^2} \qquad \text{for} \qquad |x - c| < \frac{1}{2}c.$$

Therefore, for these values of x we have

$$\left|\varphi(x) - \frac{1}{c}\right| \le \frac{2}{c^2}|x - c|. \tag{2}$$

In order to make this last term less than ε it suffices to take $|x - c| < \frac{1}{2}c^2\varepsilon$. Consequently, if we choose

$$\delta(\varepsilon) := \inf\left\{\frac{1}{2}c,\ \frac{1}{2}c^2\varepsilon\right\},$$

then if $0 < |x - c| < \delta(\varepsilon)$, it will follow first that $|x - c| < \frac{1}{2}c$ so that (2) is valid, and therefore, since $|x - c| < \left(\frac{1}{2}c^2\right)\varepsilon$, that

$$\left|\varphi(x) - \frac{1}{c}\right| = \left|\frac{1}{x} - \frac{1}{c}\right| < \varepsilon.$$

Since we have a way of choosing $\delta(\varepsilon) > 0$ for an arbitrary choice of $\varepsilon > 0$, we infer that $\lim_{x\to c} \varphi = 1/c$.

(e) $\lim_{x\to 2} \frac{x^3 - 4}{x^2 + 1} = \frac{4}{5}$.

Let $\psi(x) := (x^3 - 4)/(x^2 + 1)$ for $x \in \mathbb{R}$. Then a little algebraic manipulation gives us

$$\begin{aligned}\left|\psi(x) - \frac{4}{5}\right| &= \frac{|5x^3 - 4x^2 - 24|}{5(x^2 + 1)} \\ &= \frac{|5x^3 + 6x + 12|}{5(x^2 + 1)} \cdot |x - 2|.\end{aligned}$$

To get a bound on the coefficient of $|x-2|$, we restrict x by the condition $1 < x < 3$. For x in this interval, we have $5x^2 + 6x + 12 \le 5 \cdot 3^2 + 6 \cdot 3 + 12 = 75$ and $5(x^2+1) \ge 5(1+1) = 10$, so that

$$\left|\psi(x) - \frac{4}{5}\right| \le \frac{75}{10}|x-2| = \frac{15}{2}|x-2|.$$

Now for given $\varepsilon > 0$, we choose

$$\delta(\varepsilon) := \inf\left\{1, \frac{2}{15}\varepsilon\right\}.$$

Then if $0 < |x-2| < \delta(\varepsilon)$, we have $|\psi(x) - (4/5)| \le (15/2)|x-2| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, the assertion is proved. □

Sequential Criterion for Limits

Theorem *(Sequential Criterion) Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. Then the following are equivalent.*

(i) $\lim_{x \to c} f = L$.

(ii) *For every sequence (x_n) in A that converges to c such that $x_n \ne c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to L.*

Proof. (i) $\Rightarrow$ (ii). Assume f has limit L at c, and suppose (x_n) is a sequence in A with $\lim(x_n) = c$ and $x_n \ne c$ for all n. We must prove that the sequence $(f(x_n))$ converges to L. Let $\varepsilon > 0$ be given. Then by Definition 4.1.4, there exists $\delta > 0$ such that if $x \in A$ satisfies

$0 < |x - c| < \delta$, then $f(x)$ satisfies $|f(x) - L| < \varepsilon$. We now apply the definition of convergent sequence for the given δ to obtain a natural number $K(\delta)$ such that if $n > K(\delta)$ then $|x_n - c| < \delta$. But for each such x_n we have $|f(x_n) - L| < \varepsilon$. Thus if $n > K(\delta)$, then $|f(x_n) - L| < \varepsilon$. Therefore, the sequence $(f(x_n))$ converges to L.

(ii) $\Rightarrow$ (i). [The proof is a contrapositive argument.] If (i) is not true, then there exists an ε_0-neighborhood $V_{\varepsilon_0}(L)$ such that no matter what δ-neighborhood of c we pick, there will be at least one number x_δ in $A \cap V_\delta(c)$ with $x_\delta \ne c$ such that $f(x_\delta) \notin V_{\varepsilon_0}(L)$. Hence for every $n \in \mathbb{N}$, the $(1/n)$-neighborhood of c contains a number x_n such that

$$0 < |x_n - c| < 1/n \qquad \text{and} \qquad x_n \in A,$$

but such that

$$|f(x_n) - L| \ge \varepsilon_0 \qquad \text{for all} \qquad n \in \mathbb{N}.$$

We conclude that the sequence (x_n) in $A \backslash \{c\}$ converges to c, but the sequence $(f(x_n))$ does not converge to L. Therefore we have shown that if (i) is not true, then (ii) is not true. We conclude that (ii) implies (i). Q.E.D.

Divergence Criteria *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A.*

(a) *If $L \in \mathbb{R}$, then f does **not** have limit L at c if and only if there exists a sequence (x_n) in A with $x_n \ne c$ for all $n \in \mathbb{N}$ such that the sequence (x_n) converges to c but the sequence $(f(x_n))$ does **not** converge to L.*

(b) *The function f does **not** have a limit at c if and only if there exists a sequence (x_n) in A with $x_n \ne c$ for all $n \in \mathbb{N}$ such that the sequence (x_n) converges to c but the sequence $(f(x_n))$ does **not** converge in $\mathbb{R}$.*

Examples **(a)** $\lim_{x\to 0}(1/x)$ does not exist in $\mathbb{R}$.

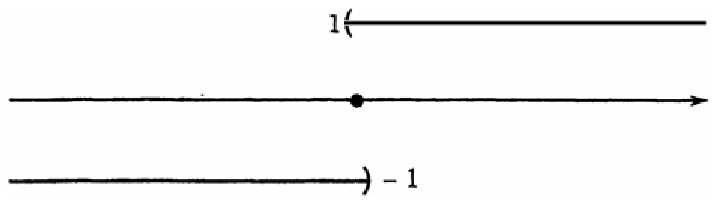
As in Example 4.1.7(d), let $\varphi(x) := 1/x$ for $x > 0$. However, here we consider $c = 0$. The argument given in Example 4.1.7(d) breaks down if $c = 0$ since we cannot obtain a bound such as that in (2) of that example. Indeed, if we take the sequence (x_n) with $x_n := 1/n$ for $n \in \mathbb{N}$, then $\lim(x_n) = 0$, but $\varphi(x_n) = 1/(1/n) = n$. As we know, the sequence $(\varphi(x_n)) = (n)$ is not convergent in $\mathbb{R}$, since it is not bounded. Hence, by Theorem 4.1.9(b), $\lim_{x\to 0}(1/x)$ does not exist in $\mathbb{R}$.

(b) $\lim_{x\to 0}\operatorname{sgn}(x)$ does not exist.

Let the **signum function** sgn be defined by

$$\operatorname{sgn}(x) := \begin{cases} +1 & \text{for} \quad x > 0, \\ 0 & \text{for} \quad x = 0, \\ -1 & \text{for} \quad x < 0. \end{cases}$$

Note that $\operatorname{sgn}(x) = x/|x|$ for $x \neq 0$. (See Figure 4.1.2.) We shall show that sgn does not have a limit at $x = 0$. We shall do this by showing that there is a sequence (x_n) such that $\lim(x_n) = 0$, but such that $(\operatorname{sgn}(x_n))$ does not converge.



The signum function

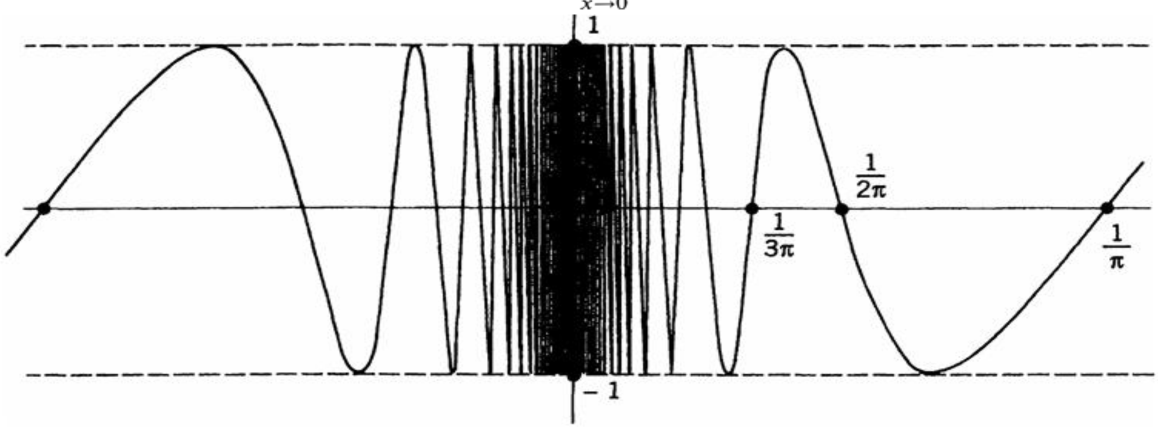
Indeed, let $x_n := (-1)^n/n$ for $n \in \mathbb{N}$ so that $\lim(x_n) = 0$. However, since

$$\operatorname{sgn}(x_n) = (-1)^n \quad \text{for} \quad n \in \mathbb{N},$$

it follows from Example 3.4.6(a) that $(\operatorname{sgn}(x_n))$ does not converge. Therefore $\lim_{x\to 0}\operatorname{sgn}(x)$ does not exist.

(c)† $\lim_{x\to 0}\sin(1/x)$ does not exist in $\mathbb{R}$.

Let $g(x) := \sin(1/x)$ for $x \neq 0$. (See Figure 4.1.3.) We shall show that g does not have a limit at $c = 0$, by exhibiting two sequences (x_n) and (y_n) with $x_n \neq 0$ and $y_n \neq 0$ for all $n \in \mathbb{N}$ and such that $\lim(x_n) = 0$ and $\lim(y_n) = 0$, but such that $\lim(g(x_n)) \neq \lim(g(y_n))$. In view of Theorem 4.1.9 this implies that $\lim_{x\to 0} g$ cannot exist. (Explain why.)



The function $g(x) = \sin(1/x)(x \neq 0)$

Indeed, we recall from calculus that $\sin t = 0$ if $t = n\pi$ for $n \in \mathbb{Z}$, and that $\sin t = +1$ if $t = \frac{1}{2}\pi + 2\pi n$ for $n \in \mathbb{Z}$. Now let $x_n := 1/n\pi$ for $n \in \mathbb{N}$; then $\lim(x_n) = 0$ and $g(x_n) = \sin n\pi = 0$ for all $n \in \mathbb{N}$, so that $\lim(g(x_n)) = 0$. On the other hand, let $y_n := \left(\frac{1}{2}\pi + 2\pi n\right)^{-1}$ for $n \in \mathbb{N}$; then $\lim(y_n) = 0$ and $g(y_n) = \sin\left(\frac{1}{2}\pi + 2\pi n\right) = 1$ for all $n \in \mathbb{N}$, so that $\lim(g(y_n)) = 1$. We conclude that $\lim_{x\to 0} \sin(1/x)$ does not exist. □

Limit Theorems

Definition Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. We say that f is **bounded on a neighborhood of** c if there exists a δ-neighborhood $V_\delta(c)$ of c and a constant $M > 0$ such that we have $|f(x)| \le M$ for all $x \in A \cap V_\delta(c)$.

Theorem *If $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ has a limit at $c \in \mathbb{R}$, then f is bounded on some neighborhood of c.*

Proof. If $L := \lim_{x\to c} f$, then for $\varepsilon = 1$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < 1$; hence (by Corollary 2.2.4(a)),

$$|f(x)| - |L| \le |f(x) - L| < 1.$$

Therefore, if $x \in A \cap V_\delta(c)$, $x \ne c$, then $|f(x)| \le |L| + 1$. If $c \notin A$, we take $M = |L| + 1$, while if $c \in A$ we take $M := \sup\{|f(c)|, |L| + 1\}$. It follows that if $x \in A \cap V_\delta(c)$, then $|f(x)| \le M$. This shows that f is bounded on the neighborhood $V_\delta(c)$ of c. Q.E.D.

Definition Let $A \subseteq \mathbb{R}$ and let f and g be functions defined on A to $\mathbb{R}$. We define the **sum** $f + g$, the **difference** $f - g$, and the **product** fg on A to $\mathbb{R}$ to be the functions given by

$$(f+g)(x) := f(x) + g(x), \qquad (f-g)(x) := f(x) - g(x),$$
$$(fg)(x) := f(x)g(x)$$

for all $x \in A$. Further, if $b \in \mathbb{R}$, we define the **multiple** bf to be the function given by

$$(bf)(x) := bf(x) \quad \text{for all} \quad x \in A.$$

Finally, if $h(x) \ne 0$ for $x \in A$, we define the **quotient** f/h to be the function given by

$$\left(\frac{f}{h}\right)(x) := \frac{f(x)}{h(x)} \qquad \text{for all} \qquad x \in A.$$

Theorem *Let $A \subseteq \mathbb{R}$, let f and g be functions on A to $\mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. Further, let $b \in \mathbb{R}$.*

(a) *If $\lim_{x\to c} f = L$ and $\lim_{x\to c} g = M$, then:*

$$\lim_{x\to c}(f+g) = L + M, \qquad \lim_{x\to c}(f-g) = L - M,$$
$$\lim_{x\to c}(fg) = LM, \qquad \lim_{x\to c}(bf) = bL.$$

(b) *If $h : A \to \mathbb{R}$, if $h(x) \ne 0$ for all $x \in A$, and if $\lim_{x\to c} h = H \ne 0$, then*

$$\lim_{x\to c}\left(\frac{f}{h}\right) = \frac{L}{H}.$$

Proof. One proof of this theorem is exactly similar to that of Theorem 3.2.3. Alternatively, it can be proved by making use of Theorems 3.2.3 and 4.1.8. For example, let (x_n) be any sequence in A such that $x_n \neq c$ for $n \in \mathbb{N}$, and $c = \lim(x_n)$. It follows from Theorem 4.1.8 that

$$\lim(f(x_n)) = L, \qquad \lim(g(x_n)) = M.$$

On the other hand, Definition 4.2.3 implies that

$$(fg)(x_n) = f(x_n)g(x_n) \qquad \text{for} \quad n \in \mathbb{N}.$$

Therefore an application of Theorem 3.2.3 yields

$$\begin{aligned} \lim((fg)(x_n)) &= \lim(f(x_n)g(x_n)) \\ &= [\lim(f(x_n))]\,[\lim(g(x_n))] = LM. \end{aligned}$$

Consequently, it follows from Theorem 4.1.8 that

$$\lim_{x\to c}(fg) = \lim((fg)(x_n)) = LM.$$

The other parts of this theorem are proved in a similar manner. We leave the details to the reader. Q.E.D.

Remark Let $A \subseteq \mathbb{R}$, and let $f_1, f_2, \dots, f_n$ be functions on A to $\mathbb{R}$, and let c be a cluster point of A. If $L_k := \lim_{x\to c} f_k$ for $k = 1, \dots, n$, then it follows from Theorem 4.2.4 by an Induction argument that

$$L_1 + L_2 + \cdots + L_n = \lim_{x\to c}(f_1 + f_2 + \cdots + f_n),$$

and

$$L_1 \cdot L_2 \cdots L_n = \lim(f_1 \cdot f_2 \cdots f_n).$$

In particular, we deduce that if $L = \lim_{x\to c} f$ and $n \in \mathbb{N}$, then

$$L^n = \lim_{x\to c}(f(x))^n.$$

(a) Some of the limits that were established in Section 4.1 can be proved by using Theorem 4.2.4. For example, it follows from this result that since $\lim_{x\to c} x = c$, then $\lim_{x\to c} x^2 = c^2$, and that if $c > 0$, then

$$\lim_{x\to c}\frac{1}{x} = \frac{1}{\lim_{x\to c} x} = \frac{1}{c}.$$

(b) $\lim_{x\to 2}\left(x^2+1\right)\left(x^3-4\right) = 20.$

It follows from Theorem 4.2.4 that

$$\begin{aligned} \lim_{x\to 2}(x^2+1)(x^3-4) &= \left(\lim_{x\to 2}(x^2+1)\right)\left(\lim_{x\to 2}(x^3-4)\right) \\ &= 5\cdot 4 = 20. \end{aligned}$$

(c) $\lim_{x\to 2}\left(\dfrac{x^3-4}{x^2+1}\right) = \dfrac{4}{5}.$

$$\lim_{x\to 2}\frac{x^3-4}{x^2+1} = \frac{\lim\limits_{x\to 2}(x^3-4)}{\lim\limits_{x\to 2}(x^2+1)} = \frac{4}{5}.$$

Note that since the limit in the denominator [i.e., $\lim\limits_{x\to 2}(x^2+1) = 5$] is not equal to 0, then Theorem 4.2.4(b) is applicable.

(d) $\lim\limits_{x\to 2} = \dfrac{x^2-4}{3x-6} = \dfrac{4}{3}$.

If we let $f(x) := x^2 - 4$ and $h(x) := 3x - 6$ for $x \in \mathbb{R}$, then we *cannot* use Theorem 4.2.4(b) to evaluate $\lim\limits_{x\to 2}(f(x)/h(x))$ because

$$H = \lim_{x\to 2} h(x) = \lim_{x\to 2}(3x-6) = 3\cdot 2 - 6 = 0.$$

However, if $x \neq 2$, then it follows that

$$\frac{x^2-4}{3x-6} = \frac{(x+2)(x-2)}{3(x-2)} = \frac{1}{3}(x+2).$$

Therefore we have

$$\lim_{x\to 2}\frac{x^2-4}{3x-6} = \lim_{x\to 2}\frac{1}{3}(x+2) = \frac{1}{3}\left(\lim_{x\to 2} x + 2\right) = \frac{4}{3}.$$

Note that the function $g(x) = (x^2-4)/(3x-6)$ has a limit at $x = 2$ *even though it is not defined there.*

(e) $\lim\limits_{x\to 0}\dfrac{1}{x}$ does not exist in $\mathbb{R}$.

Of course $\lim\limits_{x\to 0} 1 = 1$ and $H := \lim\limits_{x\to 0} x = 0$. However, since $H = 0$, we *cannot* use Theorem 4.2.4(b) to evaluate $\lim\limits_{x\to 0}(1/x)$. In fact, as was seen in Example 4.1.10(a), the function $\varphi(x) = 1/x$ does not have a limit at $x = 0$. This conclusion also follows from Theorem 4.2.2 since the function $\varphi(x) = 1/x$ is not bounded on a neighborhood of $x = 0$.

(f) If p is a polynomial function, then $\lim\limits_{x\to c} p(x) = p(c)$.

Let p be a polynomial function on $\mathbb{R}$ so that $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ for all $x \in \mathbb{R}$. It follows from Theorem 4.2.4 and the fact that $\lim\limits_{x\to c} x^k = c^k$ that

$$\begin{aligned}
\lim_{x\to c} p(x) &= \lim_{x\to c}\left[a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\right] \\
&= \lim_{x\to c}(a_n x^n) + \lim_{x\to c}\left(a_{n-1}x^{n-1}\right) + \cdots + \lim_{x\to c}(a_1 x) + \lim_{x\to c} a_0 \\
&= a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0 \\
&= p(c).
\end{aligned}$$

Hence $\lim\limits_{x\to c} p(x) = p(c)$ for any polynomial function p.

(g) If p and q are polynomial functions on $\mathbb{R}$ and if $q(c) \neq 0$, then

$$\lim_{x\to c}\frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}.$$

(g) If p and q are polynomial functions on $\mathbb{R}$ and if $q(c) \neq 0$, then

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}.$$

Since $q(x)$ is a polynomial function, it follows from a theorem in algebra that there are at most a finite number of real numbers $\alpha_1, \ldots, \alpha_m$ [the real zeroes of $q(x)$] such that $q(\alpha_j) = 0$ and such that if $x \notin \{\alpha_1, \ldots, \alpha_m\}$, then $q(x) \neq 0$. Hence, if $x \notin \{\alpha_1, \ldots, \alpha_m\}$, we can define

$$r(x) := \frac{p(x)}{q(x)}.$$

If c is not a zero of $q(x)$, then $q(c) \neq 0$, and it follows from part (f) that $\lim_{x \to c} q(x) = q(c) \neq 0$. Therefore we can apply Theorem 4.2.4(b) to conclude that

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{\lim_{x \to c} p(x)}{\lim_{x \to c} q(x)} = \frac{p(c)}{q(c)}.$$

□

Theorem *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. If*

$$a \leq f(x) \leq b \quad \textit{for all} \quad x \in A, x \neq c,$$

and if $\lim_{x \to c} f$ exists, then $a \leq \lim_{x \to c} f \leq b$.

Proof. Indeed, if $L = \lim_{x \to c} f$, then it follows from Theorem 4.1.8 that if (x_n) is any sequence of real numbers such that $c \neq x_n \in A$ for all $n \in \mathbb{N}$ and if the sequence (x_n) converges to c, then the sequence $(f(x_n))$ converges to L. Since $a \leq f(x_n) \leq b$ for all $n \in \mathbb{N}$, it follows from Theorem 3.2.6 that $a \leq L \leq b$. Q.E.D.

Squeeze Theorem *Let $A \subseteq \mathbb{R}$, let $f, g, h: A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. If*

$$f(x) \leq g(x) \leq h(x) \quad \textit{for all} \quad x \in A,\ x \neq c,$$

and if $\lim_{x \to c} f = L = \lim_{x \to c} h$, then $\lim_{x \to c} g = L$.

Examples **(a)** $\lim_{x \to 0} x^{3/2} = 0\ (x > 0)$.

Let $f(x) := x^{3/2}$ for $x > 0$. Since the inequality $x < x^{1/2} \leq 1$ holds for $0 < x \leq 1$ (why?), it follows that $x^2 \leq f(x) = x^{3/2} \leq x$ for $0 < x \leq 1$. Since

$$\lim_{x \to 0} x^2 = 0 \quad \text{and} \quad \lim_{x \to 0} x = 0,$$

it follows from the Squeeze Theorem 4.2.7 that $\lim_{x \to 0} x^{3/2} = 0$.

(b) $\lim_{x \to 0} \sin x = 0$.

It will be proved later (see Theorem 8.4.8), that

$$-x \leq \sin x \leq x \quad \text{for all} \quad x \geq 0.$$

Since $\lim_{x \to 0} (\pm x) = 0$, it follows from the Squeeze Theorem that $\lim_{x \to 0} \sin x = 0$.

(c) $\lim_{x \to 0} \cos x = 1.$

It will be proved later (see Theorem 8.4.8) that

$$(1) \qquad 1 - \tfrac{1}{2}x^2 \le \cos x \le 1 \quad \text{for all} \quad x \in \mathbb{R}.$$

Since $\lim_{x \to 0} \left(1 - \frac{1}{2}x^2\right) = 1$, it follows from the Squeeze Theorem that $\lim_{x \to 0} \cos x = 1$.

(d) $\lim_{x \to 0} \left(\dfrac{\cos x - 1}{x}\right) = 0.$

We cannot use Theorem 4.2.4(b) to evaluate this limit. (Why not?) However, it follows from the inequality (1) in part (c) that

$$-\tfrac{1}{2}x \le (\cos x - 1)/x \le 0 \quad \text{for} \quad x > 0$$

and that

$$0 \le (\cos x - 1)/x \le -\tfrac{1}{2}x \quad \text{for} \quad x < 0.$$

Now let $f(x) := -x/2$ for $x \ge 0$ and $f(x) := 0$ for $x < 0$, and let $h(x) := 0$ for $x \ge 0$ and $h(x) := -x/2$ for $x < 0$. Then we have

$$f(x) \le (\cos x - 1)/x \le h(x) \quad \text{for} \quad x \ne 0.$$

Since it is readily seen that $\lim_{x \to 0} f = 0 = \lim_{x \to 0} h$, it follows from the Squeeze Theorem that $\lim_{x \to 0} (\cos x - 1)/x = 0$.

(e) $\lim_{x \to 0} \left(\dfrac{\sin x}{x}\right) = 1.$

Again we cannot use Theorem 4.2.4(b) to evaluate this limit. However, it will be proved later (see Theorem 8.4.8) that

$$x - \tfrac{1}{6}x^3 \le \sin x \le x \quad \text{for} \quad x \ge 0$$

and that

$$x \le \sin x \le x - \tfrac{1}{6}x^3 \quad \text{for} \quad x \le 0.$$

Therefore it follows (why?) that

$$1 - \tfrac{1}{6}x^2 \le (\sin x)/x \le 1 \quad \text{for all} \quad x \ne 0.$$

But since $\lim_{x \to 0} \left(1 - \frac{1}{6}x^2\right) = 1 - \frac{1}{6} \cdot \lim_{x \to 0} x^2 = 1$, we infer from the Squeeze Theorem that $\lim_{x \to 0} (\sin x)/x = 1$.

(f) $\lim_{x \to 0} (x \sin(1/x)) = 0.$

Let $f(x) = x \sin(1/x)$ for $x \ne 0$. Since $-1 \le \sin z \le 1$ for all $z \in \mathbb{R}$, we have the inequality

$$-|x| \le f(x) = x \sin(1/x) \le |x|$$

for all $x \in \mathbb{R}, x \ne 0$. Since $\lim_{x \to 0} |x| = 0$, it follows from the Squeeze Theorem that $\lim_{x \to 0} f = 0$.

Theorem *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A. If*

$$\lim_{x \to c} f > 0 \quad \left[\textit{respectively}, \ \lim_{x \to c} f < 0\right],$$

then there exists a neighborhood $V_\delta(c)$ of c such that $f(x) > 0$ [respectively, $f(x) < 0$] for all $x \in A \cap V_\delta(c)$, $x \neq c$.

Proof. Let $L := \lim_{x \to c} f$ and suppose that $L > 0$. We take $\varepsilon = \frac{1}{2}L > 0$ in Definition 4.1.4, and obtain a number $\delta > 0$ such that if $0 < |x - c| < \delta$ and $x \in A$, then $|f(x) - L| < \frac{1}{2}L$. Therefore (why?) it follows that if $x \in A \cap V_\delta(c)$, $x \neq c$, then $f(x) > \frac{1}{2}L > 0$.

If $L < 0$, a similar argument applies. Q.E.D.

One-Sided Limits

There are times when a function f may not possess a limit at a point c, yet a limit does exist when the function is restricted to an interval on one side of the cluster point c.

Definition Let $A \in \mathbb{R}$ and let $f : A \to \mathbb{R}$.

(i) If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (c, \infty) = \{x \in A: x > c\}$, then we say that $L \in \mathbb{R}$ is a **right-hand limit of f at c** and we write

$$\lim_{x \to c+} f = L \quad \text{or} \quad \lim_{x \to c+} f(x) = L$$

if given any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for all $x \in A$ with $0 < x - c < \delta$, then $|f(x) - L| < \varepsilon$.

(ii) If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (-\infty, c) = \{x \in A: x < c\}$, then we say that $L \in \mathbb{R}$ is a **left-hand limit of f at c** and we write

$$\lim_{x \to c-} f = L \quad \text{or} \quad \lim_{x \to c-} f(x) = L$$

if given any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in A$ with $0 < c - x < \delta$, then $|f(x) - L| < \varepsilon$.

Theorem *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of $A \cap (c, \infty)$. Then the following statements are equivalent:*

(i) $\lim_{x \to c+} f = L$.

(ii) *For every sequence (x_n) that converges to c such that $x_n \in A$ and $x_n > c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to L.*

Theorem *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of both of the sets $A \cap (c, \infty)$ and $A \cap (-\infty, c)$. Then $\lim_{x \to c} f = L$ if and only if $\lim_{x \to c+} f = L = \lim_{x \to c-} f$.*

Infinite Limits

Definition Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A.

(i) We say that f **tends to** ∞ **as** $x \to c$, and write

$$\lim_{x \to c} f = \infty,$$

if for every $\alpha \in \mathbb{R}$ there exists $\delta = \delta(\alpha) > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta$, then $f(x) > \alpha$.

(ii) We say that f **tends to** $-\infty$ **as** $x \to c$, and write

$$\lim_{x \to c} f = -\infty,$$

if for every $\beta \in \mathbb{R}$ there exists $\delta = \delta(\beta) > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta$, then $f(x) < \beta$.

Examples **(a)** $\lim_{x \to 0} (1/x^2) = \infty$.

For, if $\alpha > 0$ is given, let $\delta := 1/\sqrt{\alpha}$. It follows that if $0 < |x| < \delta$, then $x^2 < 1/\alpha$ so that $1/x^2 > \alpha$.

(b) Let $g(x) := 1/x$ for $x \neq 0$.

The function g does *not* tend to either ∞ or $-\infty$ as $x \to 0$. For, if $\alpha > 0$ then $g(x) < \alpha$ for all $x < 0$, so that g does not tend to ∞ as $x \to 0$. Similarly, if $\beta < 0$ then $g(x) > \beta$ for all $x > 0$, so that g does not tend to $-\infty$ as $x \to 0$. □

Theorem *Let $A \subseteq \mathbb{R}$, let $f, g : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. Suppose that $f(x) \leq g(x)$ for all $x \in A, x \neq c$.*

(a) *If $\lim_{x \to c} f = \infty$, then $\lim_{x \to c} g = \infty$.*

(b) *If $\lim_{x \to c} g = -\infty$, then $\lim_{x \to c} f = -\infty$.*

Proof. (a) If $\lim_{x \to c} f = \infty$ and $\alpha \in \mathbb{R}$ is given, then there exists $\delta(\alpha) > 0$ such that if $0 < |x - c| < \delta(\alpha)$ and $x \in A$, then $f(x) > a$. But since $f(x) \leq g(x)$ for all $x \in A, x \neq c$, it follows that if $0 < |x - c| < \delta(\alpha)$ and $x \in A$, then $g(x) > \alpha$. Therefore $\lim_{x \to c} g = \infty$.

The proof of (b) is similar. Q.E.D.

Definition Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (c, \infty) = \{x \in A : x > c\}$, then we say that f **tends to** ∞ [respectively, $-\infty$] as $x \to c+$, and we write

$$\lim_{x \to c+} f = \infty \quad \left[\text{respectively, } \lim_{x \to c+} f = -\infty\right],$$

if for every $\alpha \in \mathbb{R}$ there is $\delta = \delta(\alpha) > 0$ such that for all $x \in A$ with $0 < x - c < \delta$, then $f(x) > \alpha$ [respectively, $f(x) < \alpha$].

Examples **(a)** Let $g(x) := 1/x$ for $x \neq 0$.

$\lim_{x \to 0} g$ does not exist. However, it is an easy exercise to show that

$$\lim_{x \to 0+} (1/x) = \infty \quad \text{and} \quad \lim_{x \to 0-} (1/x) = -\infty.$$

Limits at Infinity

It is also desirable to define the notion of the limit of a function as $x \to \infty$. The definition as $x \to -\infty$ is similar.

Definition Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. We say that $L \in \mathbb{R}$ is a **limit of** f **as** $x \to \infty$, and write

$$\lim_{x\to\infty} f = L \quad \text{or} \quad \lim_{x\to\infty} f(x) = L,$$

if given any $\varepsilon > 0$ there exists $K = K(\varepsilon) > a$ such that for any $x > K$, then $|f(x) - L| < \varepsilon$.

Theorem *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. Then the following statements are equivalent:*

(i) $L = \lim_{x\to\infty} f$.
(ii) *For every sequence (x_n) in $A \cap (a, \infty)$ such that $\lim(x_n) = \infty$, the sequence $(f(x_n))$ converges to L.*

We leave it to the reader to prove this theorem and to formulate and prove the companion result concerning the limit as $x \to -\infty$.

Examples **(a)** Let $g(x) := 1/x$ for $x \neq 0$.
It is an elementary exercise to show that $\lim_{x\to\infty}(1/x) = 0 = \lim_{x\to-\infty}(1/x)$.

(b) Let $f(x) := 1/x^2$ for $x \neq 0$.
The reader may show that $\lim_{x\to\infty}(1/x^2) = 0 = \lim_{x\to-\infty}(1/x^2)$.

Definition Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Suppose that $(a, \infty) \subseteq A$ for some $a \in A$. We say that f **tends to** ∞ [respectively, $-\infty$] **as** $x \to \infty$, and write

$$\lim_{x\to\infty} f = \infty \quad \left[\text{respectively, } \lim_{x\to\infty} f = -\infty\right]$$

if given any $\alpha \in \mathbb{R}$ there exists $K = K(\alpha) > a$ such that for any $x > K$, then $f(x) > \alpha$ [respectively, $f(x) < \alpha$].

Theorem *Let $A \in \mathbb{R}$, let $f : A \to \mathbb{R}$, and suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. Then the following statements are equivalent:*

(i) $\lim_{x\to\infty} f = \infty$ *[respectively,* $\lim_{x\to\infty} f = -\infty$*].*
(ii) *For every sequence (x_n) in (a, ∞)such that $\lim(x_n) = \infty$, then $\lim(f(x_n)) = \infty$ [respectively, $\lim(f(x_n)) = -\infty$].*

Theorem *Let $A \subseteq \mathbb{R}$, let $f, g : A \to \mathbb{R}$, and suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. Suppose further that $g(x) > 0$for all $x > a$ and that for some $L \in \mathbb{R}$, $L \neq 0$, we have*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L.$$

(i) *If $L > 0$, then $\lim_{x\to\infty} f = \infty$ if and only if $\lim_{x\to\infty} g = \infty$.*
(ii) *If $L < 0$, then $\lim_{x\to\infty} f = -\infty$ if and only if $\lim_{x\to\infty} g = \infty$.*

Proof. (i) Since $L > 0$, the hypothesis implies that there exists $a_1 > a$ such that

$$0 < \frac{1}{2}L \leq \frac{f(x)}{g(x)} < \frac{3}{2}L \qquad \text{for} \qquad x > a_1.$$

Therefore we have $\left(\frac{1}{2}L\right)g(x) < f(x) < \left(\frac{3}{2}L\right)g(x)$ for all $x > a_1$, from which the conclusion follows readily.

The proof of (ii) is similar. Q.E.D.

4.3.16 Examples **(a)** $\lim_{x\to\infty} x^n = \infty$ for $n \in \mathbb{N}$.

Let $g(x) := x^n$ for $x \in (0, \infty)$. Given $\alpha \in \mathbb{R}$, let $K := \sup\{1, \alpha\}$. Then for all $x > K$, we have $g(x) = x^n \geq x > \alpha$. Since $\alpha \in \mathbb{R}$ is arbitrary, it follows that $\lim_{x\to\infty} g = \infty$.

(b) $\lim_{x\to-\infty} x^n = \infty$ for $n \in \mathbb{N}$, n even, and $\lim_{x\to-\infty} x^n = -\infty$ for $n \in \mathbb{N}$, n odd.

We will treat the case n odd, say $n = 2k + 1$ with $k = 0, 1, \ldots$. Given $\alpha \in \mathbb{R}$, let $K := \inf\{\alpha, -1\}$. For any $x < K$, then since $(x^2)^k \geq 1$, we have $x^n = (x^2)^k x \leq x < \alpha$. Since $\alpha \in \mathbb{R}$ is arbitrary, it follows that $\lim_{x\to-\infty} x^n = -\infty$.

(c) Let $p : \mathbb{R} \to \mathbb{R}$ be the polynomial function

$$p(x) := a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Then $\lim_{x\to\infty} p = \infty$ if $a_n > 0$, and $\lim_{x\to\infty} p = -\infty$ if $a_n < 0$.

Indeed, let $g(x) := x^n$ and apply Theorem 4.3.15. Since

$$\frac{p(x)}{g(x)} = a_n + a_{n-1}\left(\frac{1}{x}\right) + \cdots + a_1\left(\frac{1}{x^{n-1}}\right) + a_0\left(\frac{1}{x^n}\right),$$

it follows that $\lim_{x\to\infty}(p(x)/g(x)) = a_n$. Since $\lim_{x\to\infty} g = \infty$, the assertion follows from Theorem 4.3.15.

(d) Let p be the polynomial function in part (c). Then $\lim_{x\to-\infty} p = \infty$ [respectively, $-\infty$] if n is even [respectively, odd] and $a_n > 0$.

We leave the details to the reader. □

References.
1.Intermediate Real Analysis by Emanuel Fischer
2.A Basic Course in Real Analysis by Ajit Kumar
3.Introduction of Real analysis by G.R Bartle