UNIT I

LAPLACE TRANSFORMS

1. Introduction :

A transformation is mathematical operations, which transforms a mathematical expressions into another equivalent simple form. For example, the transformation logarithms converts multiplication division, powers into simple addition, subtraction and multiplication respectively.

The Laplace transform is one which enables us to solve differential equation by use of algebraic methods. Laplace transform is a mathematical tool which can be used to solve many problems in Science and Engineeing. This transform was first introduced by Laplace, a French mathematician, in the year 1790, in his work on probability theory. This technique became very popular when heaveside funcitons was applied ot the solution of ordinary differential equation in electrical Engeneering problems.

Many kinds of transformation exist, but Laplace transform and fourier transform are the most well known. The Laplace transform is related to fourier transform, but whereas the fourier transform expresses a function or signal as a series of mode of vibrations, the Laplace transform resolves a function into its moments.

Like the fourier transfrom, the Laplace transform is used for solving differential and integral equations. In Physics and Engineering it is used for analysis of linear time invariant systems such as electrical circuits, harmonic oscillators, optical devices and mechanical systems. In such analysis, the Laplace transform is often interpreted as a transformation form the time domain in which inputs and outputs are functions of time, to the frequency domain, where the same inputs and outputs are functions of complex angular frequency in radius per unit time. Given a simple mathematical or functional discription of an input or output to a system, the Laplace transform provides an alternative functional discription that often simplifies the process of analyzing the behaviour of the system or in synthesizing a new system based on a set of specification. The Laplace transform belongs to the family of integral transforms. The solutions of mechanical or electrical problems involving discontinuous force function are obtained easily by Laplace transforms.

1.1 DEFINITION OF LAPLACE TRANSFORMS

Let $f(t)$ be a functions of the variable t which is defined for all positive values of t. Let s be the real constant.

If the integral $\int_0^\infty e^{-st} f(t)dt$ exist and is equal to F(s), then F(s) is called the Laplace transform of $f(t)$ and is denoted by the symbol $L[f(t)]$.

$$\text{i.e } L[f(t)] = \int_0^\infty e^{-st} f(t)dt = F\left[s\right]$$

The Laplace Transform of $f(t)$ is said to exist if the integral converges for some values of s, otherwise it does not exist.

Here the operator L is called the Laplace transform operator which transforms the functions $f(t)$ into F(s).

Remark : $\underset{S\to\infty}{L\text{im}} F(s) = 0$

1.2. Piecewise continuous function :

A function $f(t)$ is said to be piecewise continuous in any interval $[a, b]$ if it is defined on that interval, and the interval can be divided into a finite number of sub intervals in each of which $f(t)$ is continuous.

In otherwords piecewise continuous means $f(t)$ can have only finite numer of finite discontinuities.

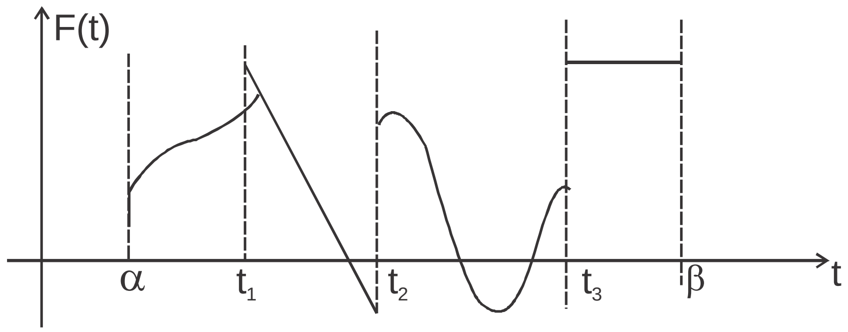


Figure 1.1

An example of a function which is periodically or sectional continuous is shown graphically in Fig 1.1. above. This function has discontinuities at t_1, t_2 and t_3.

1.3. Definition of Exponential order:

A function $f(t)$ is said to be of exponential order if $\underset{t\to\infty}{Lim} e^{-st} f(t) = 0$.

1.4. Sufficient conditions for the existence of the Laplace Transforms :

Let $f(t)$ be defined and continuous for all positive values of t. The Laplace Transform of $f(t)$ exists if the following conditions are satisfied.

(i) $f(t)$ is piecewise continuous (or) sectionally continuous.

(ii) $f(t)$ should be of exponential order.

1.5. Seven Indeterminates

1.	$\frac{0}{0}$	4.	$\infty \times \infty$	7.	0^0
2.	$\frac{\infty}{\infty}$	5.	1^∞		
3.	$0 \times \infty$	6.	∞^0		

Example:

Check whether the following functions are exponential or not $(a)\ f(t) = t^2 \quad (b)\ f(t) = e^{t^2}$

Solution :

$(a)\ f(t) = t^2$

By the definition of exponential order

$$\underset{t\to\infty}{Lim} e^{-st} f(t) = 0$$

$$\Rightarrow \underset{t\to\infty}{Lim} e^{-st} \cdot t^2$$

$$\Rightarrow \underset{t\to\infty}{Lim} \frac{t^2}{e^{st}} \Rightarrow \left(\frac{\infty}{\infty}\right)$$ which is indeterminate form

Apply L - Hospital Rule

$$\lim_{t\to\infty} \frac{2t}{e^{st} \times s} \Rightarrow \left(\frac{\infty}{\infty}\right)$$ which is indeterminate form

Again apply L - Hospital Rule.

$$\Rightarrow \lim_{t\to\infty} \frac{2}{s^2 e^{st}} \Rightarrow \lim_{t\to\infty} \frac{2}{s^2} \cdot e^{-st} = 0$$ (finite)

$$\therefore \lim_{t\to\infty} e^{-st} \cdot t^2 = 0$$ (finite numbers)

Hence $f(t) = t^2$ is exponential order.

(b) $f(t) = e^{t^2}$

Solution :

By the definition of exponential order.

$$\Rightarrow \lim_{t\to\infty} \; e^{-st} f(t) = 0$$

$$\Rightarrow \lim_{t\to\infty} \; e^{-st} \cdot e^{t^2} \Rightarrow \lim_{t\to\infty} \; e^{-st+t^2} = e^{\infty} = \infty$$

$\therefore$ $f(t) = e^{t^2}$ is not of exponential order.

2. Laplace Transform of Standard functions :

(1) Prove that $L[e^{-at}] = \dfrac{1}{s+a}$ where $s + a > 0$ or $s > -a$

Proof :

By definition $L[f(t)] = \int_0^\infty e^{-st} f(t) dt$

$$L[e^{-at}] = \int_0^\infty e^{-st} \cdot e^{-at} dt$$

$$= \int_0^\infty e^{-t(s+a)} dt$$

$$= \left[\frac{-e^{-t(s+a)}}{s+a}\right]_0^\infty = \frac{-1}{s+a}\left[e^{-\infty} - e^0\right]$$

$$= \frac{1}{s+a}$$

Hence $L[e^{-at}] = \dfrac{1}{s+a}$

2. Prove that $L[e^{at}] = \frac{1}{s-a}$ where $s > a$

Proof :

By the defn of $L[f(t)] = \int_0^\infty e^{-st} f(t)dt$

$$L[e^{+at}] = \int_0^\infty e^{-st} \cdot e^{at} dt$$

$$= \int_0^\infty e^{-(s-a)t} dt$$

$$= \left[\frac{-e^{-(s-a)t}}{s-a} \right]_0^\infty$$

$$= \frac{-1}{s-a}\left[e^{-\infty} - e^0 \right]$$

$$= \frac{1}{s-a}$$

Hence $L[e^{at}] = \frac{1}{s-a}$

3. $$L(\cos at) = \int_0^\infty e^{-st} \cos at \ dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2} (-s\cos at + a\sin at) \right]_0^\infty$$

$$= 0 - \frac{1}{s^2+a^2}(-s)$$

$$= \frac{s}{s^2+a^2}$$

$$\because \int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx \right]$$

$$\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx \right]$$

Hence $L(\cos at) = \frac{s}{s^2+a^2}$

4. $$L(\sin at) = \int_0^\infty e^{-st} \sin at \ dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2} (-s\sin at - a\cos at) \right]_0^\infty$$

$$= 0 - \frac{1}{s^2+a^2}(0-a)$$

$$L(\sin at) = \frac{a}{s^2+a^2}$$

5. $L(\cos hat) = \frac{1}{2} L(e^{at} + e^{-at})$

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{1}{2}\left(\frac{s+a+s-a}{(s+a)(s-a)}\right)$$

$$= \frac{s}{s^2 - a^2}$$

$$L(\cos hat) = \frac{s}{s^2 - a^2}$$

6. $L(\sin hat) = \frac{1}{2} L(e^{at} - e^{-at})$

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right)$$

$$= \frac{1}{2}\left(\frac{(s+a)-(s-a)}{(s-a)(s+a)}\right)$$

$$= \frac{a}{s^2 - a^2}$$

$$L(\sin hat) = \frac{a}{s^2 - a^2}$$

7. $L(1) = \int_0^\infty e^{-st} \cdot 1 \cdot dt$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \left(0 - \frac{1}{-s}\right) = \frac{1}{s}$$

$$L(1) = \frac{1}{s}$$

8.. $L(t^n) = \int_0^{\infty} e^{-St} t^n dt$

$$= \left[(t^n) \frac{e^{-St}}{-s} \right]_0^{\infty} - \int_0^{\infty} n t^{n-1} \left(\frac{e^{-St}}{-s} \right) dt$$

$$= (0-0) + \frac{n}{s} \int_0^{\infty} e^{-St} t^{n-1} dt$$

$$= \frac{n}{s} L(t^{n-1})$$

$$L(t^n) = \frac{n}{s} L(t^{n-1})$$

$$L(t^{n-1}) = \frac{n-1}{s} L(t^{n-2})$$

$$L(t^3) = \frac{3}{s} L(t^2)$$

$$L(t^2) = \frac{2}{s} L(t)$$

$$L(t^n) = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdots\cdots \frac{3}{s} \cdot \frac{2}{s} \cdot \frac{1}{s} \cdot L(1)$$

$$= \frac{n!}{s^n} L[1] = \frac{n!}{s^n} \cdot \frac{1}{s}$$

$$L(t^n) = \frac{n!}{s^{n+1}} \, or \, \frac{\overline{|(n+1)}}{s^{n+1}}$$

In particular $n = 1, 2, 3.....$

we get $L(t) = \frac{1}{s^2}$

$$L(t^2) = \frac{2!}{s^3}$$

$$L(t^3) = \frac{3!}{s^4}$$

2.1. Linear property of Laplace Transform

1. $L(f(t) \pm g(t)) = L(f(t) \pm L(g(t))$

2. $L(Kf(t)) = KL(f(t))$

Proof (1) : By the defn of L.T

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

$$L[f(t) \pm g(t)] = \int_0^\infty e^{-st} [f(t) \pm g(t)]dt$$

$$= \int_0^\infty e^{-st} f(t)dt \pm \int_0^\infty e^{-st} g(t)dt$$

$$= L[f(t)] \pm L[g(t)]$$

Hence $L[f(t) \pm g(t)] = L[f(t)] \pm L[g(t)]$

(2) $L[Kf(t)] = KL[f(t)]$

By the defn of L.T

$$L[Kf(t)] = \int_0^\infty e^{-st} Kf(t)dt$$

$$= K\int_0^\infty e^{-st} f(t)dt$$

$$= KL[f(t)]$$

Hence $L[Kf(t)] = KL[f(t)]$

2.2. Recall

1. $2\sin A\cos B = \sin(A+B) + \sin(A-B)$
2. $2\cos A\sin B = \sin(A+B) - \sin(A-B)$
3. $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
4. $2\sin A\sin B = \cos(A-B) - \cos(A+B)$
5. $\sin^2 A = \dfrac{1-\cos 2A}{2}$
6. $\cos^2 A = \dfrac{1+\cos 2A}{2}$
7. $\sin 3A = 3\sin A - 4\sin^3 A$
8. $\cos 3A = 4\cos^3 A - 3\cos A$
9. $\sin(A+B) = \sin A\cos B + \cos A\sin B$
10. $\sin(A-B) = \sin A\cos B - \cos A\sin B$
11. $\cos(A-B) = \cos A\cos B + \sin A\sin B$
12. $\cos(A+B) = \cos A\cos B - \sin A\sin B$

3.1 Problems :

1. Find Laplace Transform of $\sin^2 t$

Solution :

$$L(\sin^2 t) = L\left(\frac{1-\cos 2t}{2}\right)$$

$$= \frac{1}{2} L(1-\cos 2t)$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

2. Find $L(\cos^3 t)$

Solution :

we know that $\cos 3A = 4\cos^3 A - 3\cos A$

hence $\cos^3 A = \frac{3}{4}\cos A + \frac{1}{4}\cos 3A$

$$L(\cos^3 t) = \frac{1}{4} L(3\cos t + \cos 3t)$$

$$= \frac{1}{4}\left(\frac{3s}{s^2+1} + \frac{s}{s^2+9}\right)$$

3. Find $L(\sin 3t \cos t)$

Solution :

we know that $\sin A \cos B = \frac{1}{2}\big(\sin(A+B) + \sin(A-B)\big)$

hence $\sin 3t \cos t = \frac{1}{2}(\sin 4t + \sin 2t)$

$$L(\sin 3t \cos t) = \frac{1}{2} L(\sin 4t + \sin 2t)$$

$$= \frac{1}{2}\left(\frac{4}{s^2+16} + \frac{2}{s^2+4}\right)$$

$$= \frac{2}{s^2+16} + \frac{1}{s^2+4}$$

4. Find $L(\sin t \sin 2t \sin 3t)$

Solution :

we know that $\sin t \sin 2t \sin 3t = \sin t \frac{1}{2}(\cos t - \cos 5t)$

$$= \frac{1}{2}\sin t \cos t - \frac{1}{2}(\sin t \cos 5t)$$

$$= \frac{1}{4}\sin 2t - \frac{1}{4}(\sin 6t - \sin 4t)$$

$$L(\sin t \sin 2t \sin 3t) = \frac{1}{4} L(\sin 2t + \sin 4t - \sin 6t)$$

$$= \frac{1}{4}\left[\frac{2}{s^2+4} + \frac{4}{s^2+16} - \frac{6}{s^2+36}\right]$$

5. Find $L(1+e^{-3t}-5e^{4t})$

Solution :

$$L\left[1+e^{-3t}-5e^{4t}\right]=L[1]L\left[e^{-3t}\right]+5L\left[e^{4t}\right]$$
$$=\frac{1}{s}+\frac{1}{s+3}-\frac{5}{s-4}$$

6. Find $L(3+e^{6t}+\sin 2t-5\cos 3t)$

Solution :

$$L(3+e^{6t}+\sin 2t-5\cos 3t)=3L(1)+L(e^{6t})+L(\sin 2t)-5L(\cos 3t)$$
$$=3\cdot\frac{1}{s}+\frac{1}{s-6}+\frac{2}{s^2+4}-\frac{5s}{s^2+9}$$

7. Find $L(\sin(2t+3))$

Solution :

$$L(\sin(2t+3))=L(\sin 2t\cos 3+\sin 3\cos 2t)$$
$$=\cos 3L(\sin 2t)+\sin 3L(\cos 2t)$$
$$=\cos 3\frac{2}{s^2+4}+\sin 3\frac{s}{s^2+4}$$

8. Find $L(\sin 4t+3\sin h2t-4\cos h5t+e^{-5t})$

Solution :

$$L(\sin 4t+3\sin h2t-4\cos h5t+e^{-5t})$$
$$=L(\sin 4t)+3L(\sin h2t)-4L(\cos h5t)+L(e^{-5t})$$
$$=\frac{4}{s^2+16}+3\cdot\frac{2}{s^2-4}-4\cdot\frac{s}{s^2-25}+\frac{1}{s+5}$$
$$=\frac{4}{s^2+16}+\frac{6}{s^2-4}-\frac{4s}{s^2-25}+\frac{1}{s+5}$$

9. Find $L((1+t)^2)$

Solution :

$$L((1+t)^2)=L(1+2t+t^2)$$
$$=L(1)+2L(t)+L(t^2)$$
$$=\frac{1}{s}+2\cdot\frac{1}{s^2}+\frac{2!}{s^3}$$

10. Find the Laplace Transform of $f(t)=\begin{cases}\sin t & 0<t<\pi \\ 0 & t>\pi\end{cases}$

Solution :

By definition,

$$L(f(t))=\int_0^\infty e^{-st}f(t)dt$$

$$=\int_0^\pi e^{-st}f(t)dt+\int_\pi^\infty e^{-st}f(t)dt$$

$$=\int_0^\pi e^{-st}\sin t\ dt+\int_\pi^\infty e^{-st}(0)dt$$

$$=\int_0^\pi e^{-st}\sin t\ dt$$

$$=\left[\frac{e^{-st}}{(-s)^2+1^2}(-s\sin t-\cos t)\right]_0^\pi \qquad \because \int e^{ax}\sin bx\ dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)$$

$$=\frac{e^{-s\pi}}{s^2+1}(-s\sin\pi-\cos\pi)-\frac{e^0}{s^2+1}(0-1)$$

$$=\frac{e^{-s\pi}}{s^2+1}(1)+\frac{1}{s^2+1}$$

$$=\frac{1}{s^2+1}(e^{-s\pi}+1)$$

11. Find the Laplace Transform of $f(t)=\begin{cases}e^t & 0<t<1 \\ 0 & t>1\end{cases}$

Solution :

By definition, $$L(f(t))=\int_0^\infty e^{-st}f(t)dt$$

$$=\int_0^1 e^{-st}f(t)dt+\int_1^\infty e^{-st}f(t)dt$$

$$=\int_0^1 e^{-st}e^t dt+\int_1^\infty e^{-st}0dt$$

$$=\int_0^1 e^{(-s+1)t}dt$$

$$=\left[\frac{e^{(1-s)t}}{1-s}\right]_0^1$$

$$=\frac{1}{1-s}\left(e^{1-s}-1\right)$$

3.2. Note :

1. $\Gamma(n+1)=\int_0^\infty x^n e^{-x}dx$ (By definition)

$\Gamma(n+1)=n!,\quad n=1,2,3,.....$

$\Gamma(n+1)=n\Gamma(n),\quad n>0$

12. Find $L\left(\frac{1}{\sqrt{t}}+t^{3/2}\right)$

Solution :

$$L\left(\frac{1}{\sqrt{t}}+t^{3/2}\right)=L(t^{-1/2})+L(t^{3/2})$$

$$=\frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}}+\frac{\Gamma\left(\frac{3}{2}+1\right)}{s^{\frac{3}{2}+1}}$$

$$=\frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}+\frac{3}{2}\cdot\frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{5}{2}}}$$

$$=\frac{\sqrt{\pi}}{\sqrt{s}}+\frac{3}{4}\frac{\sqrt{\pi}}{s^{\frac{5}{2}}}$$

4. First Shifting Theorem (First translation)

1. If $L(f(t))=F(s)$, then $L(e^{-at}f(t))=F(s+a)$

Proof :

By definition, $L\left[f(t)\right]=\int_0^\infty e^{-st}f(t)dt$

$$L\left[e^{-at}f(t)\right]=\int_0^\infty e^{-st}\cdot e^{-at}f(t)dt$$

$$=\int_0^\infty e^{-t(s+a)}f(t)dt$$

$$=F(s+a)$$

Hence $L\left[e^{-at}f(t)\right]=F(s+a)$

4.1. Corollary : $L(e^{at}f(t))=F(s-a)$

4.2. Note :

1. $L(e^{-at}f(t))=L\left[f(t)\right]_{s\to s+a}$

$$=\left[F(s)\right]_{s\to s+a}$$

$$=F(s+a)$$

2. $L(e^{at}f(t))=L\left[f(t)\right]_{s\to s-a}$

$$=\left[F(s)\right]_{s\to s-a}$$

$$=F(s-a)$$

4.3. Problems :

1. Find $L(te^{2t})$

Solution :

$$L(te^{2t}) = \left[L(t)\right]_{s\to s-2}$$

$$= \left(\frac{1}{s^2}\right)_{s\to s-2} = \frac{1}{(s-2)^2}$$

2. Find $L(t^5e^{-t})$

Solution :

$$L(t^5e^{-t}) = \left[L(t^5)\right]_{s\to s+1}$$

$$= \left(\frac{5!}{s^6}\right)_{s\to s+1}$$

$$= \frac{5!}{(s+1)^6}$$

3. Find $L(e^{-2t}\sin 3t)$

Solution :

$$L(e^{-2t}\sin 3t) = L(\sin 3t)_{s\to s+2}$$

$$= \left(\frac{3}{s^2+9}\right)_{s\to s+2}$$

$$= \frac{3}{(s+2)^2+9}$$

4. Find $L(e^{-t}\cos h4t)$

Solution :

$$L(e^{-t}\cos h4t) = L(\cos h4t)_{s\to s+1}$$

$$= \left(\frac{s}{s^2-16}\right)_{s\to s+1}$$

$$= \frac{s+1}{(s+1)^2-16}$$

5. Find $L(e^{3t}\sin^2 4t)$

Solution :

$$L(e^{3t}\sin^2 4t) = L(\sin^2 4t)_{s\to s-3}$$

$$= L\left(\frac{1-\cos 8t}{2}\right)_{s\to s-3}$$

$$= \frac{1}{2}\left(L(1) - L(\cos 8t)\right)_{s\to s-3}$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+64}\right)_{S\to S-3}$$

$$= \frac{1}{2}\left(\frac{1}{s-3} - \frac{s-3}{(s-3)^2+64}\right)$$

6. Find $L(e^{-2t}\sin 4t\cos 6t)$

Solution :

$$L(e^{-2t}\sin 4t\cos 6t)=L(\sin 4t\cos 6t)_{s\to s+2}$$
$$=\frac{1}{2}\left(L(2\sin 4t\cos 6t)\right)_{s\to s+2}$$
$$=\frac{1}{2}\left(L(\sin(4t+6t)+(\sin 4t-6t)\right)_{s\to s+2}$$
$$=\frac{1}{2}\left(L(\sin 10t-\sin 2t)\right)_{s\to s+2}$$
$$=\frac{1}{2}\left(\frac{10}{s^2+100}-\frac{2}{s^2+4}\right)_{s\to s+2}$$
$$=\frac{1}{2}\left(\frac{10}{(s+2)^2+100}-\frac{2}{(s+2)^2+4}\right)$$

7. Find $L\left(e^{4t}(\sin^3 3t+\cosh^3 3t)\right)$

Solution:

$$L\left(e^{4t}(\sin^3 3t+\cosh^3 3t)\right)=L\left(\sin^3 3t+\cosh^3 3t\right)_{s\to s-4}$$
$$=L\left(\frac{3\sin 3t-\sin 9t}{4}+\frac{3\cosh 3t+\cosh 9t}{4}\right)_{s\to s-4}$$
$$\because \sin^3\theta=\frac{3\sin\theta-\sin 3\theta}{4},\cosh^3\theta=\frac{3\cosh\theta+\cosh 3\theta}{4}$$
$$=\left[\frac{3}{4}L(\sin 3t)-\frac{1}{4}L(\sin 9t)+\frac{3}{4}L(\cosh 3t)+\frac{1}{4}(\cosh 9t)\right]_{s\to s-4}$$
$$=\left(\frac{3}{4}\cdot\frac{3}{s^2+9}-\frac{1}{4}\frac{9}{s^2+81}+\frac{3}{4}\cdot\frac{s}{s^2-9}+\frac{1}{4}\frac{s}{s^2-81}\right)_{s\to s-4}$$
$$=\frac{3}{4}\cdot\frac{3}{(s-4)^2+9}-\frac{1}{4}\frac{9}{(s-4)^2+81}+\frac{3}{4}\cdot\frac{s-4}{(s-4)^2-9}+\frac{1}{4}\frac{s-4}{(s-4)^2-81}$$

8. Find $L(\cosh t\cos 2t)$

Solution :

$$L(\cos ht\cos 2t)=L\left(\left(\frac{e^t+e^{-t}}{2}\right)\cos 2t\right)$$
$$=\frac{1}{2}L(e^t\cos 2t+e^{-t}\cos 2t)$$
$$=\frac{1}{2}\left[L(\cos 2t)_{s\to s-1}+L(\cos 2t)_{s\to s+1}\right]$$
$$=\frac{1}{2}\left[\left(\frac{s}{s^2+4}\right)_{s\to s-1}+\left(\frac{s}{s^2+4}\right)_{s\to s+1}\right]$$
$$=\frac{1}{2}\left(\frac{s-1}{(s-1)^2+4}+\frac{s+1}{(s+1)^2+4}\right)$$

5. Theorem

If $L(f(t)) = F(s)$, then $L(tf(t)) = \frac{-d}{ds}(F(s))$

Proof :

Given $F(s) = L(f(t))$

differentiate both sides, w.r. to 's'

$$\frac{d}{ds}(F(s)) = \frac{d}{ds}(L(f(t)))$$

$$= \frac{d}{ds}\left(\int_0^\infty e^{-st} f(t)dt\right)$$

$$= \int_0^\infty \frac{\partial}{\partial s}(e^{-st} f(t))dt$$

$$= \int_0^\infty (-t)e^{-st} f(t)dt$$

$$= -\int_0^\infty tf(t)e^{-st}dt$$

$$\frac{d}{ds}(F(s)) = -L(tf(t))$$

$$\therefore L(tf(t)) = \frac{-d}{ds}F(s)$$

(or) $L(tf(t)) = -F'(s)$ where $F(s) = L(f(t))$

similarly we can show that,

$$L(t^2 f(t)) = (-1)^2 \frac{d^2}{ds^2}F(s)$$

$$L(t^3 f(t)) = (-1)^3 \frac{d^3}{ds^3}F(s)$$

In general, $L(t^n f(t)) = (-1)^n \frac{d^n}{ds^n}F(s)$

5.1. Problems :

1. Find $L(te^{3t})$

Solution :

We know that $L(tf(t)) = \frac{-d}{ds}L(f(t))$

Here $f(t) = e^{3t}$

$$L(te^{3t}) = \frac{-d}{ds} L(e^{3t})$$

$$= \frac{-d}{ds}\left(\frac{1}{s-3}\right)$$

$$= -\left(\frac{(s-3)(0)-(1)}{(s-3)^2}\right)$$

$$= \frac{1}{(s-3)^2}$$

2. Find $L(t\sin 3t)$

Solution :

$$L(tf(t)) = \frac{-d}{ds} L(f(t))$$

$$L(tf(t)) = \frac{-d}{ds} L(\sin 3t)$$

$$= \frac{-d}{ds}\left(\frac{3}{s^2+9}\right)$$

$$= \left(\frac{-(s^2+9)(0)+3(2s)}{(s^2+9)^2}\right)$$

$$= \frac{6s}{(s^2+9)^2}$$

3. Find $L(t\cos^2 3t)$

Solution :

$$L(t\cos^2 3t) = \frac{-d}{ds} L(\cos^2 3t)$$

$$= \frac{-d}{ds} L\left(\frac{1+\cos 6t}{2}\right)$$

$$= \frac{-1}{2}\frac{d}{ds}\left(L(1)+L(\cos 6t)\right)$$

$$= \frac{-1}{2}\frac{d}{ds}\left(\frac{1}{s}+\frac{s}{s^2+16}\right)$$

$$= \frac{-1}{2}\left(\frac{-1}{s^2}+\frac{(s^2+16)\cdot 1-s(2s)}{(s^2+16)^2}\right)$$

$$= \frac{-1}{2}\left(\frac{-1}{s^2}+\frac{16-s^2}{(s^2+16)^2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{s^2}+\frac{s^2-16}{(s^2+16)^2}\right)$$

4. Find $L(te^{-2t}\sin 3t)$

Solution :

$$L(e^{-2t}(t\sin 3t)) = L(t\sin 3t)_{s\to s+2}$$
$$= \left\{\frac{-d}{ds}(L(\sin 3t)\right\}_{s\to s+2}$$
$$= \left\{\frac{-d}{ds}\left(\frac{3}{s^2+9}\right)\right\}_{s\to s+2}$$
$$= \left\{\frac{(s^2+9)0-3(2s)}{(s^2+9)^2}\right\}_{s\to s+2}$$
$$= \frac{6(s+2)}{((s+2)^2+9)^2}$$

5. Find $L(te^{-2t}\sin 2t\sin 3t)$

Solution :

$$L(te^{-2t}\sin 2t\sin 3t)$$
$$= L(t\sin 2t\sin 3t)_{s\to s+2}$$
$$= \left[\frac{1}{2}\times L(t\cdot 2\sin 2t\sin 3t)\right]_{s\to s+2}$$
$$= \frac{1}{2}L\big(t(\cos(2t-3t)-\cos(2t+3t))\big)_{s\to s+2}$$
$$= \frac{1}{2}L\big(t\cos t - t\cos 5t\big)_{s\to s+2}$$
$$= \frac{1}{2}\left[\frac{-d}{ds}L(\cos t)+\frac{d}{ds}L(\cos 5t)\right]_{s\to s+2}$$
$$= \frac{1}{2}\left[\frac{-d}{ds}\left(\frac{s}{s^2+1}\right)+\frac{d}{ds}\left(\frac{s}{s^2+25}\right)\right]_{s\to s+2}$$
$$= \frac{1}{2}\left[-\left(\frac{(s^2+1)\cdot 1-s(2s)}{(s^2+1)^2}\right)+\frac{d}{ds}\left(\frac{(s^2+25)\cdot 1-s(2s)}{(s^2+25)^2}\right)\right]_{s\to s+2}$$
$$= \frac{1}{2}\left[-\left(\frac{1-s^2}{(s^2+1)^2}\right)+\left(\frac{25-s^2}{(s^2+25)^2}\right)\right]_{s\to s+2}$$
$$= \frac{1}{2}\left[\frac{s^2-1}{(s^2+1)^2}+\frac{25-s^2}{(s^2+25)^2}\right]_{s\to s+2}$$
$$= \frac{1}{2}\left[\frac{(s+2)^2-1}{\left((s+2)^2+1\right)^2}+\frac{25-(s+2)^2}{\left((s+2)^2+25\right)^2}\right]$$

6. Find $L(t^2 e^{-t}\cos h2t)$

Solution :

$$L\left(e^{-t}(t^2\cos h2t)\right) = L(t^2\cos h2t)_{S\to S+1}$$

$$= \left((-1)^2\frac{d^2}{ds^2}L(\cos h2t)\right)_{S\to S+1}$$

$$= \left(\frac{d^2}{ds^2}\left(\frac{s}{s^2-4}\right)\right)_{s\to s+1}$$

$$= \left(\frac{d}{ds}\left(\frac{(s^2-4)\cdot 1 - s(2s)}{(s^2-4)^2}\right)\right)_{s\to s+1}$$

$$= \frac{d}{ds}\left(\frac{-4-s^2}{(s^2-4)^2}\right)_{s\to s+1}$$

$$= \frac{-d}{ds}\left(\frac{4+s^2}{(s^2-4)^2}\right)_{s\to s+1}$$

$$= -\left(\frac{(s^2-4)^2(2s)-(4+s^2)2(s^2-4)\cdot(2s)}{(s^2-4)^2}\right)_{s\to s+1}$$

$$= \left(-2s(s^2-4)\left(\frac{s^2-4-2(4+s^2)}{(s^2-4)^4}\right)\right)_{s\to s+1}$$

$$= \left(\frac{-2s(s^2-4-8-2s^2)}{(s^2-4)^3}\right)_{s\to s+1}$$

$$= \left(\frac{2s(s^2+12)}{(s^2-4)^3}\right)_{s\to s+1}$$

$$= \left(\frac{2(s+1)((s+1)^2+12)}{(s+1)^2-4)^3}\right)$$

6. Theorem

If $L(f(t)) = F(s)$ and if $\underset{t\to 0}{Lt}\frac{f(t)}{t}$ exist then $L\left(\frac{f(t)}{t}\right) = \int_S^\infty e^{-St} f(t)ds$

Proof :

By definition, $F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)dt$

Integrate both sides w.r.t 'S' from $S \to \infty$

$$\int_s^\infty F(s)ds = \int_s^\infty\int_0^\infty e^{-st} f(t)dtds$$

$$= \int_0^\infty \left[\int_s^\infty e^{-st} f(t) ds \right] dt$$ (Changing the order of integration since 's' and 't' are independent variable)

$$= \int_0^\infty f(t) \left(\int_s^\infty e^{-st} ds \right) dt$$

$$= \int_0^\infty f(t) dt \left(\frac{e^{-st}}{-t} \right)_s^\infty$$

$$= \int_0^\infty f(t) dt \left(\frac{-1}{t} (0 - e^{-st}) \right)$$

$$= \int_0^\infty e^{-st} \frac{f(t)}{t} dt$$

$$= L\left(\frac{f(t)}{t} \right)$$

$$\therefore L\left(\frac{f(t)}{t} \right) = \int_s^\infty L(f(t)) ds$$

Similarly we can prove that $L\left(\frac{f(t)}{t^2} \right) = \int_s^\infty \int_s^\infty L(f(t)) ds\ ds$

In general $$L\left(\frac{f(t)}{t^n} \right) = \underbrace{\int_s^\infty \int_s^\infty \cdots \int_s^\infty}_{n \text{ times}} L(f(t)) \underbrace{ds\ ds \cdots ds}_{n \text{ times}}$$

Recall :

1. $\log(AB) = \log A + \log B$
2. $\log\left(A/B \right) = \log A - \log B$
3. $\log A^B = B \log A$
4. $\log 1 = 0$
5. $\log 0 = -\infty$
6. $\log \infty = \infty$
7. $\int \frac{1}{x} dx = \log x$
8. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$
9. $\tan^{-1}(\infty) = \frac{\pi}{2}$
10. $\cot^{-1}\left(s/a \right) = \frac{\pi}{2} - \tan^{-1}\left(s/a \right)$

Problems :

1. Find $L\left(\frac{1-e^{2t}}{t}\right)$

Solution :

$\underset{t\to 0}{Lim}\frac{1-e^{2t}}{t}=\frac{0}{0}$ (Indeterminate form)

Apply L - Hospital Rule

$\underset{t\to 0}{Lim}\frac{-2e^{2t}}{1}=-2$

$\therefore$ the given function exists in the limit $t\to 0$

$$L\left(\frac{1-e^{2t}}{t}\right)=\int_s^\infty L(1-e^{2t})ds$$

$$=\int_s^\infty \left(L(1)-L(e^{2t})\right)ds$$

$$=\int_s^\infty \left(\frac{1}{s}-\frac{1}{s-2}\right)ds$$

$$=\left(\log s-\log(s-2)\right)_s^\infty$$

$$=\left[\log\left(\frac{s}{s-2}\right)\right]_s^\infty$$

$$=\left[\log\left(\frac{s}{s\left(1-2/s\right)}\right)\right]_S^\infty=\log\left(\frac{1}{1-2/s}\right)_s^\infty$$

$$=0-\log\frac{s}{s-2}$$

$$=\log\left(\frac{s}{s-2}\right)^{-1}$$

$$=\log\left(\frac{s-2}{s}\right)$$

2. Find $L\left(\frac{1-\cos at}{t}\right)$

Solution :

$\underset{t\to 0}{Lim}\frac{1-\cos at}{t}=\frac{0}{0}$ (Indeterminate form)

Apply L - Hospital Rule.

$\underset{t\to 0}{Lim}\frac{a\sin at}{1}=0$ (finite)

$\therefore$ the given function exist in the limit $t \to 0$

$$L\left(\frac{1-\cos at}{t}\right) = \int_S^\infty L(1-\cos at)ds$$

$$= \int_s^\infty \left(L(1) - L(\cos at)\right) ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+a^2}\right) ds$$

$$= \left(\log s - \frac{1}{2}\log(s^2+a^2)\right)_s^\infty$$

$$= \left(\log s - \log(s^2+a^2)^{1/2}\right)_s^\infty$$

$$= \left(\log \frac{s}{\sqrt{s^2+a^2}}\right)_s^\infty$$

$$= \left(\log \frac{s}{s\sqrt{1+a^2/s^2}}\right)_s^\infty$$

$$= \left(\log \frac{1}{\sqrt{1+a^2/s^2}}\right)_s^\infty$$

$$= \left(\log 1 - \log \frac{s}{\sqrt{s^2+a^2}}\right)$$

$$= -\log\left(\frac{s}{\sqrt{s^2+a^2}}\right) = \log\left(\frac{s}{\sqrt{s^2+a^2}}\right)^{-1} = \log\left(\frac{\sqrt{a^2+s^2}}{s}\right)$$

3. Find $L\left(\frac{e^{-at}-e^{-bt}}{t}\right)$

Solution :

$$\lim_{t\to 0} \frac{e^{-at}-e^{-bt}}{t} = \frac{0}{0} \text{ (Indeterminate form)}$$

Apply L - Hospital Rule

$$\lim_{t\to 0} \frac{-ae^{-at}+be^{-bt}}{1} = b-a$$

$\therefore$ the given function exists in the limit $t \to 0$

$$L\left(\frac{e^{-at}-e^{-bt}}{t}\right)=\int_s^\infty L(e^{-at}-e^{-bt})ds$$

$$=\int_s^\infty\left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds$$

$$=\left[\log(s+a)-\log(s+b)\right]_s^\infty$$

$$=\left[\log\left(\frac{(s+a)}{(s+b)}\right)\right]_s^\infty$$

$$=\left[\log\left(\frac{1+a/s}{1+b/s}\right)\right]_s^\infty$$

$$=\log 1-\log\left(\frac{1+a/s}{1+b/s}\right)$$

$$=\log 1-\log\left(\frac{s+a}{s+b}\right)$$

$$=-\log\left(\frac{s+a}{s+b}\right)$$

$$=\log\left(\frac{s+a}{s+b}\right)^{-1}$$

$$=\log\left(\frac{s+b}{s+a}\right)$$

4. Find $L\left(\frac{\cos at-\cos bt}{t}\right)$

Solution :

$$\lim_{t\to 0}\frac{\cos at-\cos bt}{t}=\frac{0}{0}$$ (Indeterminate form)

Apply L - Hospital Rule

$$\lim_{t\to 0}\frac{-a\sin at+b\sin bt}{1}=0$$ (finite)

$\therefore$ the given function exists in the limit $t\to 0$

$$L\left(\frac{\cos at-\cos bt}{t}\right)=\int_s^\infty L(\cos at-\cos bt)ds$$

$$=\int_s^\infty\left(\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right)ds$$

$$= \left[\frac{1}{2}\log(s^2 + a^2) - \frac{1}{2}\log(s^2 + b^2) \right]_s^\infty$$

$$= \left[\frac{1}{2}\log\frac{(s^2 + a^2)}{(s^2 + b^2)} \right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2\left(1 + a^2/s^2\right)}{s^2\left(1 + b^2/s^2\right)} \right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{\left(1 + a^2/s^2\right)}{\left(1 + b^2/s^2\right)} \right]_s^\infty$$

$$= \frac{1}{2}\left[\log 1 - \log\left(\frac{(s^2 + a^2)}{(s^2 + b^2)} \right) \right]$$

$$= \frac{1}{2}\log\left(\frac{(s^2 + b^2)}{(s^2 + a^2)} \right)$$

5. Find $L\left(\frac{e^{at} - \cos bt}{t} \right)$

Solution :

Since $\lim\limits_{t \to 0} \frac{e^{at} - \cos bt}{t}$ exists

$$L\left(\frac{e^{at} - \cos bt}{t} \right) = \int_s^\infty L(e^{at} - \cos bt)\,ds$$

$$= \int_s^\infty \left(\frac{1}{s - a} - \frac{s}{s^2 + b^2} \right) ds$$

$$= \left[\log(s - a) - \frac{1}{2}\log(s^2 + b^2) \right]_s^\infty$$

$$= \left[\log(s - a) - \log\sqrt{s^2 + b^2} \right]_s^\infty$$

$$= \left[\log\left(\frac{s - a}{\sqrt{s^2 + b^2}} \right) \right]_s^\infty$$

$$= \left[\log\left(\frac{s\left(1 - a/s\right)}{s\sqrt{1 + b^2/s^2}} \right) \right]_s^\infty$$

$$= \log 1 - \log \frac{s-a}{\sqrt{s^2+b^2}}$$

$$= -\log \frac{s-a}{\sqrt{s^2+b^2}}$$

$$= \log\left(\frac{\sqrt{s^2+b^2}}{s-a}\right)$$

6. Find $L\left(\frac{\sin^2 t}{t}\right)$

Solution :

Since $\underset{t\to 0}{Lim}\frac{\sin^2 t}{t}$ exists

$$L\left(\frac{\sin^2 t}{t}\right) = \int_s^\infty L(\sin^2 t)ds$$

$$= \int_s^\infty L\left(\frac{1-\cos 2t}{2}\right)ds$$

$$= \frac{1}{2}\int_s^\infty \left(L(1) - L(\cos 2t)\right)ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right)ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log s - \log\sqrt{s^2+4}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s}{\sqrt{s^2+4}}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s}{s\sqrt{1+4/s^2}}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{1}{\sqrt{1+4/s^2}}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log 1 - \log\frac{1}{\sqrt{1+4/s^2}}\right]$$

$$= \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)$$

7. Find $L\left(\frac{\sin 3t \cos 2t}{t}\right)$

Solution :

$$\lim_{t\to 0}\left(\frac{\sin 3t \cos 2t}{1}\right) \text{ exists}$$

$$L\left(\frac{\sin 3t \cos 2t}{t}\right) = \int_s^\infty L(\sin 3t \cos 2t)ds$$

$$= \frac{1}{2}\int_s^\infty L(2\sin 3t \cos 2t)ds$$

$$= \frac{1}{2}\int_s^\infty L(\sin 5t + \sin t)ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{5}{s^2+25} + \frac{1}{s^2+1}\right)ds$$

$$= \frac{1}{2}\left[5\cdot\frac{1}{5}\tan^{-1} s/5 + \tan^{-1} s/1\right]_s^\infty$$

$$= \frac{1}{2}\left[\tan^{-1}\left(s/5\right) + \tan^{-1}\left(s/1\right)\right]_s^\infty$$

$$= \frac{1}{2}\left(\tan^{-1}(\infty) + \tan^{-1}(\infty) - \tan^{-1}\left(s/5\right) - \tan^{-1}\left(s/1\right)\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{2} - \tan^{-1}\left(s/5\right) - \tan^{-1}\left(s/1\right)\right)$$

$$= \frac{1}{2}\left(\pi - \tan^{-1}\left(s/5\right) - \tan^{-1} s\right)$$

8. Find $L\left(\frac{\sin at}{t}\right)$. Hence find the value of $\int_0^\infty \frac{\sin t}{t}\, dt$

Solution :

Since $\lim_{t\to 0}\frac{\sin at}{t}$ exists

$$L\left(\frac{\sin at}{t}\right) = \int_s^\infty L(\sin at)\, ds$$

$$= \int_s^\infty \frac{a}{s^2+a^2}ds$$

$$= \left(a\cdot\frac{1}{a}\tan^{-1}\frac{s}{a}\right)_s^\infty$$

$$= \left(\tan^{-1}\frac{s}{a}\right)_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}\left(\frac{s}{a}\right)$$

$$= L\left(\frac{\sin at}{t}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

Deduction :

By definition

$$\int_0^\infty e^{-St}\frac{\sin at}{t} = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

Put s = 0, *a* = 1

$$\int_0^\infty \frac{\sin t}{t}dt = \frac{\pi}{2} - \tan^{-1}(0)$$

$$= \frac{\pi}{2}$$

9. Find $L\left(\frac{\cos at}{t}\right)$

Solution :

$$\underset{t\to 0}{Lt}\frac{\cos at}{t} = \frac{1}{0} = \infty$$

$\therefore \underset{t\to 0}{Lt}\frac{\cos at}{t}$ does not exist.

Hence $L\left(\frac{\cos at}{t}\right)$ does not exist.

10. Find $L\left(\frac{e^{at}}{t}\right)$

Solution :

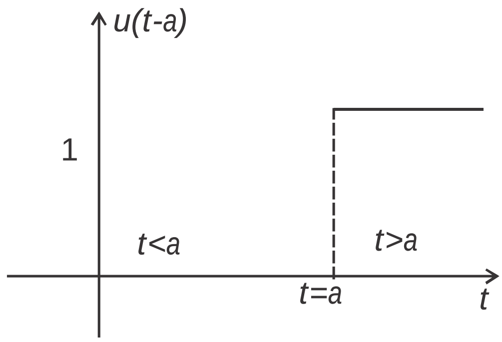
$$\underset{t\to 0}{Lt}\frac{e^{at}}{t} = \frac{1}{0} = \infty$$

$\therefore L\left(\frac{e^{at}}{t}\right)$ does not exist.

7. Unit Step function (or) heavisides unit step function :

The unit step function about the point $t = a$ is defined as $U(t-a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t > a \end{cases}$

It can also be denoted by $H(t-a)$



7.1 Find the Laplace transform of unit step function.

Solution :

The Laplace transform of unit step function is

$$L(U(t-a)) = \int_0^\infty e^{-st} U(t-a)dt$$

$$= \int_0^a e^{-st} 0 \cdot dt + \int_a^\infty e^{-st}(1)dt$$

$$= \int_a^\infty e^{-st}dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= \frac{-1}{s}(e^{-\infty} - e^{-as})$$

$$L(U(t-a)) = \frac{-1}{s}(0 - e^{-as}) = \frac{e^{-as}}{s}$$

$$\therefore L(U(t-a)) = \frac{e^{-as}}{s}$$

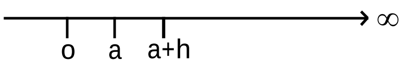
8. Dirac delta function (or) Unit Impulse function :

8.1 Dirac delta function or unit impulse function about the point t = a is defined as

$$\delta(t-a) = \begin{cases} \underset{h\to 0}{Lt} \frac{1}{h} & a < t < a+h \\ 0 & \text{otherwise} \end{cases}$$

Find the Laplace transform of Dirac delta function.

Solution :



$$L[\delta(t-a)] = \int_0^\infty e^{-st}\delta(t-a)dt$$

$$= \int_0^a e^{-st} 0 dt + \underset{h\to 0}{Lt} \frac{1}{h}\int_a^{a+h} e^{-st}dt + \int_{a+h}^\infty e^{-st} 0 dt$$

$$= \underset{h\to 0}{Lt} \frac{1}{h}\int_a^{a+h} e^{-st}dt$$

$$= \underset{h\to 0}{Lt} \frac{1}{h}\left[\frac{-1}{s}\left(e^{-(a+h)s} - e^{-as}\right)\right]$$

$$= \underset{h\to 0}{Lt} \frac{1}{h}\left[\frac{e^{-as}}{s} - \frac{e^{-(a+h)s}}{s}\right]$$

$$= \underset{h \to 0}{Lt} \frac{e^{-as}(1-e^{-hs})}{sh} = \frac{0}{0} \text{ (Indeterminate form)}$$

Applying L' Hospital Rule.

$$= \underset{h \to 0}{Lt} \frac{e^{-as}(e^{-hs}s)}{s} = e^{-as}$$

$$L(\delta(t-a)) = e^{-as} \text{ when } a = 0,\ L(\delta(t)) = 1$$

8.2. Note :

The dirac delta function is the derivative of unit step function.

9. Second shiffting Theorem (Second Translation)

If $L(f(t)) = F(s)$ and $G(t) = \begin{cases} f(t-a), & t > a \\ 0 & t < a \end{cases}$,

Then $L(G(t)) = e^{-as}F(s)$

Proof :

$$L(G(t)) = \int_0^\infty e^{-st}G(t)dt$$

$$= \int_0^a e^{-st}G(t)dt + \int_a^\infty e^{-st}G(t)dt$$

$$= \int_0^a e^{-st}0 \cdot dt + \int_a^\infty e^{-st} f(t-a)dt$$

$$= \int_a^\infty e^{-st} f(t-a)dt$$

Put $t - a = u$ when $t = a, \quad u = 0$

$dt = du$ $\quad t = \infty, \quad u = \infty$

$$\therefore L(G(t)) = \int_0^\infty e^{-s(u+a)} f(u)du$$

$$= e^{-sa}\int_0^\infty e^{-su} f(u)du$$

In $\int_0^\infty e^{-su} f(u)du$, u is a dummy variable. Hence we can replace it by the variable t.

$$\therefore L(G(t)) = e^{-sa}\int_0^\infty e^{-st} f(t)dt$$

$$= e^{-sa}L(f(t))$$

$$= e^{-as}F(S)$$

Another form of second shifting theorem

If $L(f(t)) = F(s)$ and $a > 0$ then

$L(f(t-a)U(t-a)) = e^{-as}F(s)$ where $U(t-a)$ is the unit step function.

Proof :

We know that by the definition of unit step function.

$$U(t-a) = \begin{cases} 1 & t > a \\ 0 & t < a \end{cases}$$

$$\therefore f(t-a)U(t-a) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases} \qquad \_\_\_\_(1)$$

Let $f(t-a)U(t-a) = G(t)$

$\therefore$ (1) becomes, $G(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$

which is precisely the same as the first form of second shifting theorem, as discussed above

$$\therefore L(G(t)) = e^{-as}F(s)$$

9.1. Problems :

1. Find the Laplace transform of $G(t)$, where

$$G(t) = \begin{cases} \cos\left(t - \dfrac{2\pi}{3}\right) & \text{if } t > \dfrac{2\pi}{3} \\ 0 & \text{if } t < \dfrac{2\pi}{3} \end{cases}$$

Solution :

We know that by second shifting if $L(f(t)) = F(s)$ and $G(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$

then $L(G(t)) = e^{-as}F(s)$ ____(1)

Here $f(t-a) = \cos\left(t - \dfrac{2\pi}{3}\right)$

(ie) $f(t) = \cos t \ \ \& \ \ a = \dfrac{2\pi}{3}$ ____(2)

$$\therefore L(f(t)) = L(\cos t) = \frac{s}{s^2+1} \qquad \_\_\_\_(3)$$

Substituting (2) & (3) in (1), we get

$$\therefore L(G(t)) = e^{-\frac{2\pi}{3}s} \cdot \frac{s}{s^2+1}$$

2. Find the Laplace transform using second shifting theorem for $G(t)=\begin{cases}(t-2)^3; & t>2\\ 0 & t<2\end{cases}$

Solution :

Here $a=2,\ f(t-a)=(t-2)^3$

$$f(t)=t^3$$

$$L(f(t))=L(t^3)=\frac{3!}{s^4}=F(s)$$

$$\therefore L(G(t))=e^{-as}F(s)$$

$$=e^{-2s}\frac{3!}{s^4}$$

3. Using second shifting theorem, find the Laplace transform of

$$G(t)=\begin{cases}\sin\left(t-\pi/3\right); & t>\pi/3\\ 0 & t<\pi/3\end{cases}$$

Solution :

Here $a=\frac{\pi}{3},\ f(t-a)=\sin\left(t-\pi/3\right)$

$$\therefore f(t)=\sin t$$

$$\therefore L(f(t))=L(\sin t)$$

$$=\frac{1}{s^2+1}=F(s)$$

$$\therefore L(G(t))=e^{-as}F(s)$$

$$=e^{-\pi/3 s}\cdot\frac{1}{s^2+1}$$

$$=e^{-\pi/3 s}\frac{1}{s^2+1}$$

10. Change of Scale Property

If $L(f(t))=F(s)$, Then $L(f(at))=\frac{1}{a}F\left(\frac{s}{a}\right)$

Proof :

By definition, $L(f(t))=\int_0^\infty e^{-st}f(t)dt$

$$\therefore L(f(at))=\int_0^\infty e^{-st}f(a\,t)dt$$

Put $at=y$ when $t=0,\quad y=0$

$adt=dy$ $\quad t=\infty,\quad y=\infty$

$$L(f(at)) = \int_0^\infty e^{-s(y/a)} f(y) \frac{dy}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-(s/a)y} f(y)dy$$

$$= \frac{1}{a}\int_0^\infty e^{-(s/a)t} f(t)dt$$ (Replacing the dummy variable y by t)

$$L(f(at)) = \frac{1}{a} F\left(s/a\right)$$

10.1. Corollary :

$$L\left[f\left(t/a\right)\right] = aF(as)$$

10.2. Problems :

1. Assuming $L(\sin t)$. Find $L(\sin 2t)$ and $L\left(\sin t/2\right)$

Solution :

We know that $L(\sin t) = \dfrac{1}{s^2+1}$ _____(1)

$$\therefore L(\sin 2t) = \frac{1}{2}\cdot\frac{1}{\left(s/2\right)^2+1}$$ Using (1) (Replace S by s/2)

$$L(\sin 2t) = \frac{1}{2}\left(\frac{4}{s^2+4}\right)$$

$$= \frac{2}{s^2+4}$$ _____(2)

$$\therefore L\left(\sin t/2\right) = 2\frac{1}{(2s)^2+1} = \frac{2}{4s^2+1}$$ Using (2) (Replace s by 2s)

2. Given that $L(t\cos t) = \dfrac{s^2-1}{(s^2+1)^2}$

Find (i) $L(t\cos at)$ and (ii) $L\left(t\cos\dfrac{t}{a}\right)$

Solution :

(i) Given $L(t\cos t) = \dfrac{s^2-1}{(s^2+1)^2}$

Replacing t by at

$$\therefore L(at\cos at) = \frac{1}{a}\frac{\left(\frac{s}{a}\right)^2 - 1}{\left(\left(\frac{s}{a}\right)^2 + 1\right)^2} \qquad (\because \text{ Replacing s by s/a})$$

$$L(at\cos at) = \frac{a^4(s^2 - a^2)}{a^3(s^2 + a^2)^2}$$

$$\therefore L(t\cos at) = \frac{a^4(s^2 - a^2)}{a^4(s^2 + a^2)^2} = \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

(ii) Given $L\left(t\cos\frac{t}{a}\right) = \frac{s^2 - 1}{(s^2 + 1)^2}$

Replace t by $\frac{t}{a}$, $L\left(\frac{t}{a}\cos\frac{t}{a}\right) = a\left(\frac{(as)^2 - 1}{\left((as)^2 + 1\right)^2}\right)$

$$L\left(t\cos\frac{t}{a}\right) = a^2\left(\frac{a^2s^2 - 1}{\left(a^2s^2 + 1\right)^2}\right) \qquad \text{Replace s by as.}$$

11. Laplace Transform of Derivations :

Here, we explore how the Laplace transform interacts with the basic operators of calculus differentation and integration . The greatest interest will be in the first identity that we will derive. This relates the transform of a derivative of a function to the transform of the original function, and will allow to convert many initial - value problems to easily solved algebraic Equations. But there are useful relations involving the Laplace transform and either differentiation (or) integration. So we'll look at them too.

11.1. Theorem :

If $L(f(t)) = F(s)$ Then

(i) $L(f'(t)) = sL(f(t)) - f(0)$

(ii) $L(f''(t)) = s^2L(f(t)) - sf(0) - f'(0)$

and in general

$$L(f^n(t)) = s^nL(f(t)) - s^{n-1}f(0) - s^{n-2}f'(0)......f^{n-1}(0)$$

Proof :

(i) By definition,

$$L(f'(t)) = \int_0^\infty e^{-st}f(t)dt$$

$$= \int_0^\infty e^{-st}d(f(t))$$

$$= \left(e^{-st} f(t)\right)_0^\infty - \int_0^\infty f(t) d(e^{-st})$$

$$= (0 - f(0)) - \int_0^\infty f(t) e^{-st}(-s) dt$$

$$= -f(0) + s\int_0^\infty e^{-st} f(t) dt$$

$$= -f(0) + sL(f(t))$$

$\therefore L(f'(t)) = sL(f(t)) - f(0)$ _____(1) which proves (i)

(ii) Again by definition,

$$L(f''(t)) = \int_0^\infty e^{-st} f''(t) dt$$

$$= \int_0^\infty e^{-st} d(f'(t))$$

$$= \left[e^{-st} f'(t)\right]_0^\infty - \int_0^\infty f'(t) e^{-st}(-s) dt$$

$$= \left[0 - f'(t)\right] + s\int_0^\infty e^{-st} f'(t) dt$$

$$= -f'(0) + sL(f'(t))$$

$$= sL(f'(t)) - f'(0)$$

$$= s(sL(f(t)) - f(0)) - f'(0)$$ Using (1)

$L(f''(t)) = s^2 Lf(t) - sf(0) - f'(0)$ ____(2)

Similarly proceeding like this, we can show that

$L(f^n(t)) = s^n L(f(t)) - s^{n-1} f(0) - s^{n-2} f'(0) \ldots\ldots f^{n-1}(0)$ ____(3)

The above results (1), (2) and (3) are very useful in solving linear differential equations with constant coefficients.

11.2. Note :

We have, $L(f'(t)) = sL(f(t)) - f(0)$ _____(1) and

$L(f''(t)) = s^2 L(f(t)) - sf(0) - f'(0)$ _____(2)

when $f(0) = 0$ and $f'(0) = 0$

(1) & (2) becomes

$Lf'(t) = sLf(t)$ and $Lf''(t) = s^2 Lf(t)$

This shows that under certain conditions, the process of Laplace transform replaces differentiation by multiplication by the factor s and s^2 respectively.

12. Laplace Transform of integrals

Analogous to the differentiation identities $L[f'(t)] = sF(s) - f(0)$ and $L[tf(t)] = -F'(s)$ are a pair of identities concerning transforms of integrals and integrals of transforms. These identities will not be nearly as important to us as the differentiation identities, but they do have their uses and are considered to be part of the standard set of identities for the Laplace Transform.

Before we start, however, take another look at the above differentiation identities. They show that, under the Laplace transform, the differentiation of one of the functions, f(t) or F(S) corresponds to the multiplication of the other by the approprate variable.

This may lead to suspect that the analogous integrations identities. They show that, under Laplace transform integration of one of the functions f(t) or F(S), corresponds to the division of the other by the approprate variables.

12.1. Theorem : If $L[f(t)] = F(s)$ then $L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L[f(t)]$

Proof :

Let $\int_0^t f(t)dt = \phi(t) \longrightarrow (1)$

Differentiate both sides with respect to 't'

$$\therefore f(t) = \phi'(t) \longrightarrow (2)$$

and $\phi(0) = \int_0^t f(t)dt = 0$

We know that $L[\phi(t)] = sL[\phi(t)] - \phi(0)$

$$L[\phi(t)] = sL[\phi(t)] \qquad \therefore \phi(0) = 0$$

$$\therefore L[f(t)] = sL\left[\int_0^t f(t)dt\right] \quad \text{by (1) \& (2)}$$

$$L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L[f(t)]$$

Similarly we can prove that

$$L\left[\int_0^t\int_0^t f(t)dt\right] = \frac{1}{s^2}L[f(t)]$$

$\therefore$ In general $L\left[\underbrace{\int_0^t\int_0^t \cdots \int_0^t}_{n \text{ items}} f(t)dt\right] = \frac{1}{s^n}L[f(t)]$

12.2. Note :

The above result expresses that the integral between the limits from '0' to 't' is transformed into simple division by the factor 'S' using Laplace transform.

12.3. Problems :

1. find $L\left(e^{-t}\int_0^t t\cos t\,dt\right)$

Solution :

$$L\left(e^{-t}\int_0^t t\cos t\,dt\right)=\left[L\left(\int_0^t t\cos t\,dt\right)\right]_{s\to s+1}$$

$$=\left(\frac{1}{s}L(t\cos t)\right)_{s\to s+1}$$

$$=\left(\frac{1}{s}\left(\frac{-d}{ds}(L(\cos t))\right)\right)_{s\to s+1}$$

$$=\left[\frac{1}{s}\left(\frac{-d}{ds}\left(\frac{s}{s^2+1}\right)\right)\right]_{s\to s+1}$$

$$=\left[\frac{-1}{s}\left(\frac{(s^2+1)-s(2s)}{(s^2+1)^2}\right)\right]_{s\to s+1}$$

$$=\left[\frac{-1}{s}\left(\frac{1-s^2}{(s^2+1)^2}\right)\right]_{s\to s+1}$$

$$=\left(\frac{s^2-1}{s(s^2+1)^2}\right)_{s\to s+1}$$

$$=\left(\frac{(s+1)^2-1}{(s+1)((s+1)^2+1)^2}\right)_{s\to s+1}$$

$$=\frac{s^2+2s}{(s+1)(s^2+2s+2)^2}$$

2. Find $L\left(e^{-t}\int_0^t \frac{\sin t}{t}dt\right)$

Solution :

$$L\left(e^{-t}\int_0^t \frac{\sin t}{t}dt\right)=\left[L\left(\int_0^t \frac{\sin t}{t}dt\right)\right]_{s\to s+1}$$

$$=\left[\frac{1}{s}L\left(\frac{\sin t}{t}\right)\right]_{s\to s+1}$$

Since $\lim_{t\to 0}\frac{\sin t}{t}$ exist

$$=\left[\frac{1}{s}\int_s^\infty L(\sin t)ds\right]_{s\to s+1}$$

$$=\left[\frac{1}{s}\int_s^\infty \frac{1}{s^2+1}ds\right]_{s\to s+1}$$

$$=\left[\frac{1}{s}\left(\tan^{-1}s\right)_s^\infty\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}(\tan^{-1}\infty - \tan^{-1}(s))\right]_{s \to s+1}$$

$$= \left[\frac{1}{s}\left(\frac{\pi}{2} - \tan^{-1}(s)\right)\right]_{s \to s+1}$$

$$= \left[\frac{1}{s}\cot^{-1} s\right]_{s \to s+1} \qquad = \frac{\cot^{-1}(s+1)}{s+1}$$

3. Find the Laplace Transform of $\int_0^t te^{-t} \sin t dt$

Solution :

$$L(te^{-t} \sin t dt) = (L(t \sin t))_{s \to s+1}$$

$$= \left(\frac{-d}{ds} L(\sin t)\right)_{s \to s+1}$$

$$= \left(\frac{-d}{ds}\left(\frac{1}{s^2+1}\right)\right)_{s \to s+1}$$

$$= -\left(\frac{(s^2+1)0 - 2s}{(s^2+1)^2}\right)_{s \to s+1}$$

$$= \left(\frac{2S}{(S^2+1)^2}\right)_{S \to S+1}$$

$$= \frac{2(S+1)}{((S+1)^2+1)^2}$$

$$= \frac{2(S+1)}{S^2+2S+2}$$

4. Find $L\left(\int_0^t \frac{e^{-t} \sin t}{t} dt\right)$

Solution :

$$L\left(\int_0^t \frac{e^{-t} \sin t}{t} dt\right) = \frac{1}{s} L\left(\frac{e^{-t} \sin t}{t}\right)$$

Since $\lim_{t \to 0} \frac{e^{-t} \sin t}{t}$ exist.

$$= \frac{1}{s}\left[\int_s^\infty L(e^{-t} \sin t)\right] ds$$

$$= \frac{1}{s}\left[\int_s^\infty L(\sin t)\right]_{s \to s+1} ds$$

$$= \frac{1}{s}\left[\int_s^\infty \left(\frac{1}{s^2+1}\right)\right]_{s \to s+1} ds$$

$$= \frac{1}{s}\left[\int_s^\infty \left(\frac{1}{(s+1)^2+1}\right)\right] ds$$

$$= \frac{1}{s}\left[\int_s^\infty \left(\frac{ds}{(s+1)^2+1}\right)\right]$$

$$= \frac{1}{s}\left(\tan^{-1}(s+1)\right)_s^\infty$$

$$= \frac{\cot^{-1}(s+1)}{s}$$

Problems :

1. Find $L\left(\int_0^t e^{2t} dt\right)$

Solution :

$$L\left(\int_0^t e^{2t} dt\right) = \frac{1}{s} L(e^{2t})$$

$$= \frac{1}{s} \cdot \frac{1}{s-2}$$

$$= \frac{1}{s(s-2)}$$

2. Find $L\left(\int_0^t \sin 3t dt\right)$

Solution :

$$L\left(\int_0^t \sin 3t dt\right) = \frac{1}{s} L(\sin 3t)$$

$$= \frac{1}{s} \cdot \frac{3}{s^2+9}$$

$$= \frac{3}{s(s^2+9)}$$

3. Find $L\left(\int_0^t e^{-2t} \cos 3t dt\right)$

Solution :

$$L\left(\int_0^t e^{-2t} \cos 3t dt\right) = \frac{1}{s} L(e^{-2t} \cos 3t)$$

$$= \frac{1}{s} L(\cos 3t)_{s \to s+2} \quad \text{(Using first shifting theorem)}$$

$$= \frac{1}{s}\left(\frac{s}{s^2+9}\right)_{s \to s+2}$$

$$= \frac{1}{s}\left(\frac{s+2}{(s+2)^2+9}\right)$$

4. Find $L\left(\int_0^t e^{-t}\sin h2t dt\right)$

Solution :

$$L\left(\int_0^t e^{-t}\sin h2t dt\right) = \frac{1}{s}L(e^{-t}\sin h2t)$$

$$= \frac{1}{s}L(\sin h2t)_{s\to s+1}$$

$$= \frac{1}{s}\left(\frac{2}{s^2-4}\right)_{s\to s+1}$$

$$= \frac{1}{s}\left(\frac{2}{(s+1)^2-4}\right)$$

5. Find $L\left(\int_0^t \sin 3t\cos 2t dt\right)$

Solution :

$$L\left(\int_0^t \sin 3t\cos 2t dt\right) = \frac{1}{s}L(\sin 3t\cos 2t)$$

$$= \frac{1}{2s}L(2\sin 3t\cos 2t)$$

$$= \frac{1}{2s}L(\sin 5t + \sin t)$$

$$= \frac{1}{2s}\left(\frac{5}{s^2+25} + \frac{1}{s^2+1}\right)$$

6. Find $L\left(e^{-3t}\int_0^t t\sin^2 t\right)dt$

Solution :

$$L\left(e^{-3t}\int_0^t t\sin^2 t\right) = L\left(\int_0^t t\sin^2 t dt\right)_{s\to s+3}$$

$$= \left[\frac{1}{s}L(t\sin^2 t)\right]_{s\to s+3}$$

$$= \left[\frac{-1}{s}\frac{d}{ds}L(\sin^2 t)\right]_{s\to s+3}$$

$$= \left[\frac{-1}{s}\frac{d}{ds}L\left(\frac{1-\cos 2t}{2}\right)\right]_{s\to s+3}$$

$$= \left[\frac{-1}{2s}\frac{d}{ds}L(1-\cos 2t)\right]_{s\to s+3}$$

$$= \left[\frac{-1}{2s}\frac{d}{ds}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)\right]_{s\to s+3}$$

$$= \left[\frac{-1}{2s}\left(\frac{-1}{s^2} - \frac{(s^2+4)\cdot 1 - s(2s)}{(s^2+4)^2}\right)\right]_{s\to s+3}$$

$$= \left[\frac{+1}{2s}\left(\frac{+1}{s^2} + \frac{4-s^2}{(s^2+4)^2}\right)\right]_{s\to s+3}$$

$$= \frac{1}{2(s+3)}\left(\frac{+1}{(s+3)^2} + \frac{4-(s+3)^2}{((s+3)^2+4)^2}\right)$$

$$= \frac{1}{2(s+3)^3}\left(\frac{4-(s+3)^2}{2(s+3)(s^2+6s+13)^2}\right)$$

7. Find $L\left(e^{4t}\left(\int_0^t \frac{\sin 3t \cos 2t}{t} dt\right)\right)$

Solution :

$$L\left(e^{4t}\left(\int_0^t \frac{\sin 3t \cos 2t}{t} dt\right)\right)$$

$$= L\left(\int_0^t \frac{\sin 3t \cos 2t}{t} dt\right)_{s\to s-4}$$

$$= \left[\frac{1}{s} L\left(\frac{\sin 3t \cos 2t}{t}\right)\right]_{s\to s-4}$$

$$= \left[\frac{1}{s}\int_s^\infty L(\sin 3t \cos 2t) dt\right]_{s\to s-4}$$

$$= \left[\frac{1}{2s}\int_s^\infty L(2\sin 3t \cos 2t) ds\right]_{s\to s-4}$$

$$= \left[\frac{1}{2s}\int_s^\infty L(\sin 5t + \sin t) ds\right]_{s\to s-4}$$

$$= \left[\frac{1}{2s}\int_s^\infty \left(\frac{5}{s^2+25} + \frac{1}{s^2+1}\right) ds\right]_{s\to s-4}$$

$$= \left[\frac{1}{2s}\left(5\cdot\frac{1}{5}\tan^{-1}\frac{s}{5} + \tan^{-1} s\right)\right]_{s\to s-4}$$

$$= \left[\frac{1}{2s}\left(\tan^{-1}\frac{s}{5} + \tan^{-1} s\right)_s^\infty\right]_{s\to s-4}$$

$$= \left[\frac{1}{2s}\left((\tan^{-1}\infty + \tan^{-1}\infty) - \left(\tan^{-1}\frac{s}{5} + \tan^{-1} s\right)\right)\right]_{s\to s-4}$$

$$= \left[\frac{1}{2s}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) - \tan^{-1}\frac{s}{5} - \tan^{-1} s\right]_{s\to s-4}$$

$$= \left[\frac{1}{2s}\left(\pi - \tan^{-1}\frac{s}{5} - \tan^{-1} s\right)\right]_{s\to s-4}$$

$$= \frac{1}{2(s-4)}\left(\pi - \tan^{-1}\frac{s-4}{5} - \tan^{-1}(s-4)\right)$$

13. Periodic Functions :

Laplace transform of periodic functions have a particular structure. In many applications the non homogeneous term in a linear differential equation is a periodic function. In this section, we desire a formula for the Laplace transform of such periodic functions.

13.1 Definition of Periodic functions :

A function f(t) is said to have a period T or to be periodic with period T if for all t, f(t+T)=f(t) where T is a positive constant. The least value of T>0 is called the period of f(t).

Example 1

Consider $f(t)=\sin t$

$$\begin{aligned} f(t+2\pi) &= \sin(t+2\pi) \\ &= \sin t \end{aligned}$$

(ie)
$$\begin{aligned} f(t) &= f(t+2\pi) \\ &= \sin t \end{aligned}$$

$\therefore$ $\sin t$ t is a periodic function with period 2π .

Example 2 :

$\tan t$ is a periodic function with period π .

13.2. Laplace Transform of Periodic functions :

Let $f(t)$ be a periodic function with period a.

$$f(t)=f(t+a)=f(t+2a)=f(t+3a)......$$

Now
$$\begin{aligned} L(f(t)) &= \int_0^\infty e^{-st} f(t)dt \\ &= \int_0^a e^{-st} f(t)dt + \int_a^{2a} e^{-st} f(t)dt + \int_{2a}^{3a} e^{-st} f(t)dt \\ &\quad + \int_{3a}^{4a} e^{-st} f(t)dt + \cdots\cdots \end{aligned}$$

Put in the second integral $t=T+a; \quad dt=dT$

in the Third integral $t=T+2a; \quad dt=dT$

in the Fourth integral $t=T+3a; \quad dt=dT$

When
$$\begin{aligned} t=a, &\quad T=0 \\ t=2a, &\quad T=a \end{aligned}$$

when $t = 2a, \quad T = 0$

$t = 3a, \quad T = a$

when $t = 3a, \quad T = 0$

$t = 4a, \quad T = a$

$$\therefore L(f(t)) = \int_0^a e^{-st} f(t)dt + e^{-as} \int_0^a e^{-sT} f(T+a)dT$$

$$+e^{-2as} \int_0^a e^{-sT} f(T+2a)dT + \cdots$$

$$= \int_0^a e^{-st} f(t)dt + e^{-sa} \int_0^a e^{-st} f(t+a)dt + e^{-2as} \int_0^a e^{-st} f(t+2a)dt$$

$$= (1 + e^{-as} + (e^{-as})^2 + \cdots) \int_0^a e^{-st} f(t)dt$$

$$= (1 - e^{-as})^{-1} \int_0^a e^{-st} f(t)dt \qquad \left(\because (1-x)^{-1} = 1 + x + x^2 + \cdots\right)$$

$$L(f(t)) = \frac{1}{1 - e^{-as}} \int_0^a e^{-st} f(t)dt$$

13.3. Problems :

1. Find the Laplace Transform of the square wave given by

$$f(t) = \begin{cases} E & \text{for } 0 < t < a/2 \\ -E & \text{for } a/2 < t < a \end{cases}$$

and $f(t+a) = f(t)$

Solution :

Given that $f(t+a) = f(t)$

Hence $f(t)$ is a periodic function with period $p = a$

$$L(f(t)) = \frac{1}{1 - e^{-as}} \int_0^a e^{-st} f(t)dt$$

$$= \frac{1}{1 - e^{-as}} \left[\int_0^{a/2} e^{-st} E dt + \int_{a/2}^a e^{-st}(-E)dt \right]$$

$$= \frac{1}{1 - e^{-as}} \left[E \int_0^{a/2} e^{-st} dt - E \int_{a/2}^a e^{-st} dt \right]$$

$$= \frac{E}{1-e^{-as}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{a/2} - \left(\frac{e^{-st}}{-s}\right)_{a/2}^{a}\right]$$

$$= \frac{E}{s(1-e^{-as})}\left[\left(-e^{-sa/2}+1\right)+\left(e^{-sa}-e^{-sa/2}\right)\right]$$

$$= \frac{E}{s(1-e^{-as})}\left(1-e^{-sa/2}-e^{sa/2}+e^{-sa}\right)$$

$$= \frac{E}{s(1-e^{-as})}\left(1-e^{-2sa/2}+e^{-sa}\right)$$

$$= \frac{E}{s(1-e^{-\frac{as}{2}})(1+e^{-sa/2})}\left(1-e^{\frac{-sa}{2}}\right)^2$$

$$= \frac{E\left(1-e^{\frac{-sa}{2}}\right)}{s(1+e^{-sa/2})}$$

$$= \frac{E}{s}\tan h\left(\frac{sa}{4}\right)$$

2. Find the Laplace transform of the function $f(t) = \begin{cases} t & 0 < t < b \\ 2b - t & b < t < 2b \end{cases}$

Solution :

The given function is a periodic function with period 2b

$$\therefore L(f(t)) = \frac{1}{1-e^{-2bs}}\int_0^{2b} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2bs}}\int_0^{b} e^{-st} f(t)dt + \int_b^{2b} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2bs}}\left[\int_0^{b} e^{-st} t dt + \int_b^{2b} e^{-st}(2b-t)dt\right]$$

$$= \frac{1}{1-e^{-2bs}}\left\{\begin{array}{c}\left[t\left(\frac{e^{-st}}{-s}\right)-1\left(\frac{e^{-st}}{s^2}\right)\right]_0^b + \\ \left[(2b-t)\left(\frac{e^{-st}}{-s}\right)-(-1)\left(\frac{e^{-st}}{s^2}\right)\right]_b^{2b}\end{array}\right\}$$

$$= \frac{1}{1-e^{-2bs}}\left[\frac{-be^{-sb}}{s}-\frac{e^{-sb}}{s^2}+\frac{1}{s^2}+\frac{e^{-2bs}}{s^2}+\frac{b}{s}e^{-bs}\ \frac{-e^{-bs}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2bs}}\left(\frac{1-2e^{-bs}+e^{-2bs}}{s^2}\right)$$

$$= \frac{(1-e^{-bs})^2}{s^2(1+e^{-bs})(1-e^{-bs})}$$

$$= \frac{1-e^{-bs}}{s^2(1+e^{-bs})}$$

$$= \frac{1}{s^2}\cdot\left(\frac{(1-e^{\frac{-bs}{2}})\cdot e^{\frac{-bs}{2}}}{(1+e^{\frac{-bs}{2}})\cdot e^{\frac{-bs}{2}}}\right)$$

$$= \frac{1}{s^2}\cdot\frac{e^{\frac{bs}{2}}-e^{\frac{-bs}{2}}}{e^{\frac{bs}{2}}+e^{\frac{-bs}{2}}}$$

$$= \frac{1}{s^2}\tan h\left(\frac{bs}{2}\right)$$

3. Find the Laplace transform of $f(t)=\begin{cases}\sin t & \text{in} \quad 0<t<\pi \\ 0 & \text{in} \quad \pi<t<2\pi\end{cases}$ and $f(t+2\pi)=f(t)$.

Solution :

Given that $f(t+2\pi)=f(t)$

Hence $f(t)$ is a periodic function with period $P=2\pi$

$$L(f(t)) = \frac{1}{1-e^{-sP}}\int_0^P e^{-st}f(t)dt$$

$$= \frac{1}{1-e^{-2s\pi}}\left[\int_0^{\pi} e^{-st}\sin t dt + \int_{\pi}^{2\pi} e^{-st}\cdot 0 dt\right]$$

$$= \frac{1}{1-e^{-2s\pi}}\left[\frac{1}{s^2+1}\left(e^{-st}(S\sin t - 1\cdot\cos t)\right)_0^{\pi}\right]$$

$$= \frac{1}{s^2+1}\cdot\frac{1}{1-e^{-2s\pi}}\left(e^{-s\pi}(0+1)-1(0-1)\right)$$

$$= \frac{1}{s^2+1}\frac{1}{(1-e^{-2s\pi})}\left(e^{-s\pi}+1\right)$$

$$= \frac{1}{s^2+1}\cdot\frac{1}{(1-e^{-s\pi})}\frac{\left(1+e^{-s\pi}\right)}{(1+e^{-s\pi})}$$

$$= \frac{1}{s^2+1}\cdot\frac{1}{1-e^{-s\pi}}$$

4. Find the Laplace transform of the Half-wave rectifier function

$$f(t)=\begin{cases}\sin wt, & 0<t<\frac{\pi}{w} \\ 0, & \frac{\pi}{w}<t<\frac{2\pi}{w}\end{cases}$$

Solution :

Given $f(t)=\begin{cases}\sin wt, & 0<t<\frac{\pi}{w} \\ 0, & \frac{\pi}{w}<t<\frac{2\pi}{w}\end{cases}$

This ia a periodic function with period $\frac{2\pi}{w}$ in the interval $\left(0,\frac{2\pi}{w}\right)$.

$$\therefore L(f(t)) = \frac{1}{1-e^{-\frac{2\pi s}{w}}} \int_0^{\frac{2\pi}{w}} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{w}}} \left[\int_0^{\frac{\pi}{w}} e^{-st} f(t)dt + \int_{\frac{\pi}{w}}^{\frac{2\pi}{w}} e^{-st} f(t)dt \right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{w}}} \left[\int_0^{\frac{\pi}{w}} e^{-st} \sin wt dt + \int_{\frac{\pi}{w}}^{\frac{2\pi}{w}} e^{-st} \cdot 0 dt \right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{w}}} \left[\frac{e^{-st}(-s\sin wt - w\cos wt)}{s^2+w^2} \right]_0^{\frac{\pi}{w}}$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{w}}} \left[\frac{e^{\frac{-s\pi}{w}}(w)+w}{s^2+w^2} \right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{w}}} \frac{w\left(1+e^{-s\pi/w}\right)}{s^2+w^2}$$

$$= \frac{w}{(1+e^{-s\pi/w})(1-e^{-s\pi/w})} \cdot \frac{(1+e^{-s\pi/w})}{s^2+w^2}$$

$$= \frac{w}{(1-e^{-s\pi/w})s^2+w^2}$$

5. Find the Laplace transform of the periodic function $f(t) = \begin{cases} t, & \text{for } 0<t<1 \\ 2-t, & \text{for } 1<t<2 \end{cases}$ and $f(t+2) = f(t)$

Solution :

The given function is a periodic function with period 2.

$$\therefore L(f(t)) = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} t dt + \int_1^2 (2-t)e^{-st} dt \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\left(\frac{te^{-st}}{-s} - 1 \cdot \frac{e^{-st}}{s^2} \right)_0^1 + \left((2-t)\frac{e^{-st}}{-s} + \frac{e^{-st}}{s^2} \right)_1^2 \right]$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} + \frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} \right]$$

$$= \frac{1}{1-e^{-2s}} \left(\frac{1-2e^{-s}+e^{-2s}}{s^2} \right)$$

$$= \frac{(1-e^{-s})^2}{(1-e^{-s})(1+e^{-s})s^2} = \frac{1}{s^2}\left(\frac{(1-e^{-s})}{(1+e^{-s})} \right)$$

$$= \frac{1}{s^2} \frac{e^{s/2}-e^{-s/2}}{e^{s/2}+e^{-s/2}} = \frac{1}{s^2} \tan h\left(s/2 \right)$$

6. Find the Laplace transform of the function $f(t)=\begin{cases} t, & 0<t<\frac{\pi}{2} \\ \pi-t, & \frac{\pi}{2}<t<\pi \end{cases}$ $f(\pi+t)=f(t)$

Solution :

$$\therefore L(f(t))=\frac{1}{1-e^{-s\pi}}\left[\int_0^{\pi/2} te^{-st}tdt+\int_{\pi/2}^{\pi}(\pi-t)e^{-st}dt\right]$$

$$=\frac{1}{1-e^{-s\pi}}\left[\left(\frac{te^{-st}}{-s}-\frac{e^{-st}}{s^2}\right)_0^{\pi/2}+\left((\pi-t)\frac{e^{-st}}{-s}+\frac{e^{-st}}{s^2}\right)_{\pi/2}^{\pi}\right]$$

$$=\frac{1}{1-e^{-s\pi}}\left[\frac{\pi/2\, e^{-s\pi/2}}{-s}-\frac{e^{-s\pi/2}}{s^2}+\frac{1}{s^2}+\frac{e^{-s\pi}}{s^2}-\frac{\pi/2\, e^{-s\pi/2}}{-s}+\frac{e^{-s\pi/2}}{s^2}\right]$$

$$=\frac{1}{1-e^{-s\pi}}\left[\frac{1-2e^{-s\pi/2}+e^{-s\pi}}{s^2}\right]$$

$$=\frac{(1-e^{-s\pi/2})^2}{s^2(1-e^{-s\pi/2})(1+e^{-s\pi/2})}$$

$$=\frac{1-e^{-s\pi/2}}{s^2(1+e^{-s\pi/2})}$$

7. Find the Laplace transform of the rectangular wave given by $f(t)=\begin{cases} 1, & 0<t<b \\ -1, & b<t<2b \end{cases}$

Solution :

Given $f(t)=\begin{cases} 1, & 0<t<b \\ -1, & b<t<2b \end{cases}$

This function is periodic the interval (0, 2b) with period 2b.

$$\therefore L(f(t))=\frac{1}{1-e^{-2bs}}\int_0^{2b} e^{-st}f(t)dt$$

$$=\frac{1}{1-e^{-2bs}}\left[\int_0^{b} e^{-st}f(t)dt+\int_b^{2b} e^{-st}f(t)dt\right]$$

$$=\frac{1}{1-e^{-2bs}}\left[\int_0^{b} e^{-st}(1)dt+\int_b^{2b} e^{-st}(-1)dt\right]$$

$$=\frac{1}{1-e^{-2bs}}\left[\int_0^{b} e^{-st}dt+\int_b^{2b} e^{-st}dt\right]$$

$$=\frac{1}{1-e^{-2bs}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{b}-\left(\frac{e^{-st}}{-s}\right)_b^{2b}\right]$$

$$=\frac{1}{1-e^{-2bs}}\left[\frac{e^{-sb}}{-s}+\frac{1}{s}+\frac{e^{-2sb}}{s}-\frac{e^{-sb}}{s}\right]$$

$$= \frac{1}{s}\left[\frac{1-2e^{-sb}+e^{-2sb}}{1-e^{-2bs}}\right]$$

$$= \frac{1}{s}\frac{(1-e^{-sb})^2}{(1+e^{-sb})(1-e^{-sb})}$$

$$= \frac{1}{s}\frac{1-e^{-sb}}{1+e^{-sb}}$$

$$= \frac{1}{s}\frac{(1-e^{-sb})e^{-sb/2}}{(1+e^{-sb})(e^{-sb/2})}$$

$$= \frac{1}{s}\frac{e^{sb/2}-e^{-sb/2}}{e^{sb/2}+e^{-sb/2}}$$

$$= \frac{1}{s}\tan h\left(\frac{sb}{2}\right)$$

14. Initial value theorem

If $L(f(t)) = F(s)$, then $\underset{t\to 0}{Lt}\ f(t) = \underset{t\to\infty}{Lt}\ sF(s)$

Proof :

We know that $L\left[f'(t)\right] = sL\left[f(t)\right] - f(0))$

Take the limit as $s \longrightarrow \infty$ on both sides, we have

$$\underset{s\to\infty}{Lt}\ L(f'(t)) = \underset{s\to\infty}{Lt}\,(sF(s) - f(0))$$

$$\underset{s\to\infty}{Lt}\int_0^\infty e^{-st} f'(t)\, dt = \underset{s\to\infty}{Lt}\,(sF(s) - f(0))$$ ($\because$ By definition of Laplace Transform)

$$\int_0^\infty \underset{s\to\infty}{Lt}\ e^{-st} f'(t)\, dt = \underset{s\to\infty}{Lt}\,(sF(s) - f(0))$$ ($\because$ s is independent of t, we can take the limit in the L.H.S before integration)

$$0 = \underset{s\to\infty}{Lt}\,(sF(s) - f(0))$$

$$\therefore \underset{s\to\infty}{Lt}\ sF(s) = f(0)$$

$$= \underset{t\to 0}{Lt}\ f(t)$$

$$\therefore \underset{s\to\infty}{Lt}\ sF(s) = \underset{t\to 0}{Lt}\ f(t)$$

15. Final value Theorem

If $L(f(t)) = F(s)$, then $\underset{t\to\infty}{Lt} f(t) = \underset{s\to 0}{Lt} sF(s)$

Proof :

We know that $L(f'(t)) = sL\left[f(t)\right] - f(0)$

$$L(f'(t)) = sF(s) - f(0)$$

$$\int_0^\infty e^{-st} f'(t)dt = sF(s) - f(0)$$

Take the limit as $s \to 0$ on both sides,

$$\underset{s\to 0}{Lt}\int_0^\infty e^{-st} f'(t)dt = \underset{s\to 0}{Lt}(sF(s) - f(0))$$

$$\int_0^\infty \underset{s\to 0}{Lt}\, e^{-st} f'(t)dt = \underset{s\to 0}{Lt}(sF(s) - f(0))$$ ($\because$ s is independent of t, we can take the limit in the L.H.S before integration)

$$\int_0^\infty f'(t)dt = \underset{s\to 0}{Lt}(sF(s) - f(0))$$

$$(f(t))_0^\infty = \underset{s\to 0}{Lt}(sF(s) - f(0))$$

$$\underset{t\to\infty}{Lt} f(t) - f(0) = \underset{s\to 0}{Lt} sF(s) - f(0)$$

Since $f(0)$ is n ot a function of 's' (or) 't' it can be cancelled both sides,

$$\underset{t\to\infty}{Lt} f(t) = \underset{s\to 0}{Lt} sF(s)$$

15.1.Problems :

1. If $L(f(t)) = \dfrac{1}{s(s+a)}$ find $\lim_{t\to\infty} f(t)$ and $\lim_{t\to 0} f(t)$

Solution :

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$$

$$= \lim_{s\to\infty} s \times \frac{1}{s(s+a)}$$

$$= \lim_{s\to\infty} \frac{1}{(s+a)}$$

$$= 0$$

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$= \lim_{s\to 0} s \times \frac{1}{s(s+a)}$$

$$= \lim_{s\to 0} \frac{1}{(s+a)}$$

$$= \frac{1}{a}$$

2. If $L(e^{-t}\cos^2 t) = F(s)$. Find $\lim_{s\to 0}(sF(s))$ and $\lim_{s\to\infty}(sF(s))$

Solution :

$$L(e^{-t}\cos^2 t) = F(s)$$

(ie)., $f(t) = e^{-t}\cos^2 t$

By final value theorem,

$$\lim_{s\to 0}(sF(s)) = \lim_{t\to\infty}(e^{-t}\cos^2 t) = 0$$

By initial value theorem,

$$s\lim_{S\to 0} sF(s) = \lim_{t\to 0}(e^{-t}\cos^2 t) = 1$$

3. Verify the initial and final value theorem for the function $f(t) = 1 - e^{-at}$

Solution :

Given that $f(t) = 1 - e^{-at}$ ____(1)

$$L(f(t)) = L(1 - e^{-at})$$

$$= \frac{1}{s} - \frac{1}{s+1}$$

$$F(s) = \frac{1}{s} - \frac{1}{s+1}$$

$$SF(s) = s\left(\frac{1}{s} - \frac{1}{s+a}\right)$$

$$= 1 - \frac{s}{s+a} \quad \text{\_\_\_\_(2)}$$

From (1), $\underset{t\to 0}{Lt}\, f(t) = \underset{t\to 0}{Lt}\, 1 - e^{-at}$

$$= 1 - 1$$

$$= 0 \quad \text{\_\_\_\_(3)}$$

$$\underset{t\to\infty}{Lt}\, f(t) = \underset{t\to\infty}{Lt}\, 1 - e^{-at}$$

$$= 1 - 0$$

$$= 1 \quad \text{\_\_\_\_(4)}$$

From (2), $\underset{s\to 0}{Lt}\, sF(s) = \underset{s\to 0}{Lt}\, 1 - \frac{s}{s+a} = 1$ ____(5)

$$\underset{s\to\infty}{Lt}\, sF(s) = \underset{s\to\infty}{Lt}\, 1 - \frac{s}{s+a}$$

$$= \underset{s\to\infty}{Lt}\, 1 - \frac{s}{s\left(1 + a/s\right)} = 0 \quad \text{\_\_\_\_(6)}$$

From (3) & (6), we have

$$\underset{t\to 0}{Lt}\, f(t) = \underset{s\to\infty}{Lt}\, sF(s)$$

and from (4) & (5)

$$\underset{t\to\infty}{Lt}\, f(t) = \underset{s\to 0}{Lt}\, sF(s)$$

4. Verify initial and final value theorem for the function $f(t) = e^{-2t}\cos 3t$

Solution :

Given $f(t) = e^{-2t}\cos 3t$

$$L(f(t)) = L(e^{-2t}\cos 3t)$$

$$= L(\cos 3t)_{s\to s+2}$$

$$F(s) = \left(\frac{s}{s^2+9}\right)_{s\to s+2} = \frac{s+2}{(s+2)^2+9}$$

$$SF(s) = \frac{s(s+2)}{s^2+4s+13} = \frac{s^2+2s}{s^2+4s+13}$$

$$\underset{t\to 0}{Lt}\ f(t) = \underset{t\to 0}{Lt}\ e^{-2t}\cos 3t = 1 \qquad \_\_\_\_(1)$$

$$\underset{t\to\infty}{Lt}\ f(t) = \underset{t\to\infty}{Lt}\ e^{-2t}\cos 3t = 0 \qquad \_\_\_\_(2)$$

$$\underset{s\to 0}{Lt}\ sF(s) = \underset{s\to 0}{Lt}\ \frac{s^2+2s}{s^2+4s+13} = 0 \qquad \_\_\_\_(3)$$

$$\underset{s\to\infty}{Lt}\ sF(s) = \underset{s\to\infty}{Lt}\ \frac{s^2(1+2/s)}{s^2(1+4/s+13/s^2)} = 1 \qquad \_\_\_\_(4)$$

From (1) and (4), $\underset{t\to 0}{Lt}\ f(t) = \underset{s\to\infty}{Lt}\ sF(s)$

From (2) and (3), $\underset{t\to 0}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ sF(s)$

5. Verify initial and final value theorem for $f(t) = t^2e^{-3t}$

Solution :

$$f(t) = t^2e^{-3t}$$

$$L(f(t)) = \left[L(t^2)\right]_{s\to s+3}$$

$$= \left(\frac{2!}{s^3}\right)_{s\to s+3} = \frac{2}{(s+3)^3}$$

$$sF(s) = \frac{2s}{(s+3)^3}$$

$$\underset{t\to 0}{Lt}\ f(t) = \underset{t\to 0}{Lt}\ t^2e^{-3t} = 0 \qquad \_\_\_\_(1)$$

$$\underset{t\to\infty}{Lt}\ f(t) = \underset{t\to\infty}{Lt}\ t^2e^{-3t} = 0 \qquad \_\_\_\_(2)$$

$$\underset{s\to 0}{Lt}\ sF(s) = \underset{s\to 0}{Lt}\ \frac{2s}{(s+3)^3} = 0 \qquad \_\_\_\_(3)$$

$$\underset{s\to\infty}{Lt}\ sF(s) = \underset{s\to\infty}{Lt}\ \frac{2s}{(s+3)^3} = \underset{s\to 0}{Lt}\ \frac{2s}{s^3\left(1+\frac{3}{s}\right)^3}$$

$$= \underset{s\to\infty}{Lt}\ \frac{2}{s^2\left(1+\frac{3}{s}\right)^3} = 0 \qquad \_\_\_\_(4)$$

From (1) & (4)

$$\underset{t\to 0}{Lt}\ f(t) = \underset{s\to\infty}{Lt}\ sF(s)$$

From (2) & (3)

$$\underset{t\to 0}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ sF(s).$$

UNIT II INVERSE LAPLACE TRANSFORMS

16.1. Definition :

If the Laplace transform of a function $f(t)$ is $F(S)$ (ie) $L(f(t)) = F(S)$ then $f(t)$ is called an inverse laplace transform of $F(s)$ and is denoted by

$$f(t) = L^{-1}(F(s))$$

Here L^{-1} is called the inverse Laplace transform operator.

17. Standard results in inverse Laplace transforms

Laplace Transform	Inverse Laplace Transform
$L(1) = \frac{1}{s}$	$L^{-1}\left(\frac{1}{s}\right) = 1$
$L(e^{at}) = \frac{1}{s-a}$	$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$
$L(e^{-at}) = \frac{1}{s+a}$	$L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$
$L(t) = \frac{1}{s^2}$	$L^{-1}\left(\frac{1}{s^2}\right) = t$
$L(t^2) = \frac{2!}{s^3}$	$L^{-1}\left(\frac{2!}{s^3}\right) = t^2$
$L(t^3) = \frac{3!}{s^4}$	$L^{-1}\left(\frac{3!}{s^4}\right) = t^3$
$L(t^n) = \frac{n!}{s^{n+1}}$	$L^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$
where n is a +ve integer	
$L(\sin at) = \frac{a}{s^2+a^2}$	$L^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at$
$L(\cos at) = \frac{s}{s^2+a^2}$	$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$
$L(\sin hat) = \frac{a}{s^2-a^2}$	$L^{-1}\left(\frac{a}{s^2-a^2}\right) = \sin hat$
$L(\cos hat) = \frac{s}{s^2-a^2}$	$L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cos hat$
$L(t \sin at) = \frac{2as}{(s^2+a^2)^2}$	$L^{-1}\left(\frac{2as}{(s^2+a^2)^2}\right) = t \sin at$

$$L(t\cos at)=\frac{s^2-a^2}{(s^2+a^2)^2} \qquad L^{-1}\left(\frac{s^2-a^2}{(s^2+a^2)^2}\right)=t\cos at$$

$$L(t\sin hat)=\frac{2as}{(s^2-a^2)^2} \qquad L^{-1}\left(\frac{2as}{(s^2-a^2)^2}\right)=t\sin hat$$

$$L(t\cos hat)=\frac{s^2+a^2}{(s^2-a^2)^2} \qquad L^{-1}\left(\frac{s^2+a^2}{(s^2-a^2)^2}\right)=t\cos hat$$

$$L(e^{at}\sin bt)=\frac{b}{(s-a)^2+b^2} \qquad L^{-1}\left(\frac{b}{(s-a)^2+b^2}\right)=e^{at}\sin bt$$

$$L(e^{at}\cos bt)=\frac{s-a}{(s-a)^2+b^2} \qquad L^{-1}\left(\frac{s-a}{(s-a)^2+b^2}\right)=e^{at}\cos bt$$

$$L(e^{at}\sin hbt)=\frac{b}{(s-a)^2+b^2} \qquad L^{-1}\left(\frac{b}{(s-a)^2+b^2}\right)=e^{at}\sin hbt$$

$$L(e^{at}\cos hbt)=\frac{s-a}{(s-a)^2-b^2} \qquad L^{-1}\left(\frac{s-a}{(s-a)^2-b^2}\right)=e^{at}\cos hbt$$

$$L(te^{-at})=\frac{1}{(s+a)^2} \qquad L^{-1}\left(\frac{1}{(s+a)^2}\right)=te^{-at}$$

$$L(t^2e^{-at})=\frac{2!}{(s+a)^3} \qquad L^{-1}\left(\frac{2!}{(s+a)^3}\right)=t^2e^{-at}$$

18. Properties of Inverse Laplace Transforms

18.1. Linear Property :

If $F_1(s)$ and $F_2(s)$ are Laplace transforms of $f_1(t)$ and $f_2(t)$ respectively, then

$L^{-1}(c_1F_1(s)+c_2F_2(s))=c_1L^{-1}(F_1(s))+c_2L^{-1}(F_2(s))$ where $c_1 \,\&\, c_2$ are constants.

Proof :

We know that

$$L(c_1 f_1(t)+c_2 f_2(t))=c_1L(f_1(t))+c_2L(f_2(t))$$
$$=c_1F_1(s)+c_2F_2(s)$$
$$\left[\because L(f_1(t))=F_1(s) \text{ and } L(f_2(t))=F_2(s)\right]$$

$$c_1 f_1(t)+c_2 f_2(t)=L^{-1}(c_1F_1(s)+c_2F_2(s))$$
$$=L^{-1}(c_1F_1(s))+L^{-1}(c_2F_2(s))$$
$$=c_1L^{-1}(F_1(s))+c_2L^{-1}(F_2(s))$$

Problems :

1. Find $L^{-1}\left(\frac{1}{s-3}+s+\frac{s}{s^2-4}\right)$

Solution :

$$L^{-1}\left(\frac{1}{s-3}+\frac{1}{s}+\frac{s}{s^2-4}\right)=L^{-1}\left(\frac{1}{s-3}\right)+L^{-1}(s)+L^{-1}\left(\frac{s}{s^2-4}\right)$$

$$=e^{3t}+1+\cos h2t$$

$$=e^{3t}+\cos h2t+1$$

2. Find $L^{-1}\left(\frac{1}{s^2}+\frac{1}{s+4}+\frac{1}{s^2+4}+\frac{s}{s^2-9}\right)$

Solution :

$$L^{-1}\left(\frac{1}{s^2}+\frac{1}{s+4}+\frac{1}{s^2+4}+\frac{s}{s^2-9}\right)$$

$$=L^{-1}\left(\frac{1}{s^2}\right)+L^{-1}\left(\frac{1}{s+4}\right)+L^{-1}\left(\frac{1}{s^2+4}\right)+L^{-1}\left(\frac{s}{s^2-9}\right)$$

$$=t+e^{-4t}+\frac{\sin 2t}{2}+\cos h3t$$

3. Find $L^{-1}\left(\frac{1}{s}+\frac{2}{s^2}-\frac{3s}{s^2+4}+\frac{4}{s^2+16}\right)$

Solution :

$$L^{-1}\left(\frac{1}{s}+\frac{2}{s^2}-\frac{3s}{s^2+4}+\frac{4}{s^2+16}\right)$$

$$=L^{-1}\left(\frac{1}{s}\right)+L^{-1}\left(\frac{2}{s^2}\right)-L^{-1}\left(\frac{3s}{s^2+4}\right)+L^{-1}\left(\frac{4}{s^2+16}\right)$$

$$=1+2t-3\cos 2t+\sin 4t$$

4. Find $L^{-1}\left(\frac{4}{s^6}-\frac{2}{s^{10}}+\frac{2}{s^2-9}+\frac{3s}{s^2+25}\right)$

Solution :

$$L^{-1}\left(\frac{4}{s^6}-\frac{2}{s^{10}}+\frac{2}{s^2-9}+\frac{3s}{s^2+25}\right)$$

$$=\frac{4}{5!}L^{-1}\left(\frac{5!}{s^6}\right)-\frac{2}{9!}L^{-1}\left(\frac{9!}{s^{10}}\right)+\frac{2}{3}L^{-1}\left(\frac{3}{s^2-9}\right)+3L^{-1}\left(\frac{s}{s^2+25}\right)$$

$$=\frac{1}{36}t^5-\frac{1}{181440}t^9+\frac{2}{3}\sin h3t+3\cos 5t$$

5. Find $L^{-1}\left(\frac{2}{s^5}-\frac{3}{s^4}+\frac{3}{s^2-3}+\frac{5}{s^2-100}+\frac{s}{s^2+10}\right)$

Solution :

$$L^{-1}\left(\frac{2}{s^5}-\frac{3}{s^4}+\frac{3}{s^2-3}+\frac{5}{s^2-100}+\frac{s}{s^2+10}\right)$$
$$=\frac{2}{4!}L^{-1}\left(\frac{4!}{s^5}\right)-\frac{3}{3!}L^{-1}\left(\frac{3!}{s^4}\right)+\frac{3}{\sqrt{3}}L^{-1}\left(\frac{\sqrt{3}}{s^2-\sqrt{3^2}}\right)+\frac{5}{10}L^{-1}\left(\frac{10}{s^2-100}\right)+L^{-1}\left(\frac{s}{s^2+10}\right)$$
$$=\frac{1}{12}t^4-\frac{1}{2}t^3\sqrt{3}\sin\sqrt{3}t+\frac{1}{2}\sin h10t+\cos\sqrt{10}t$$

6. Find $L^{-1}\left(\frac{5}{s^2-25}+\frac{4s}{s^2-16}+\frac{s}{s^2+9}+\frac{s}{s^2-25}\right)$

Solution :

$$L^{-1}\left(\frac{5}{s^2-25}+\frac{4s}{s^2-16}+\frac{s}{s^2+9}+\frac{s}{s^2-25}\right)$$
$$=L^{-1}\left(\frac{5}{s^2-25}\right)+4L^{-1}\left(\frac{s}{s^2-16}\right)+L^{-1}\left(\frac{s}{s^2+9}\right)+L^{-1}\left(\frac{s}{s^2-25}\right)$$
$$=\sin h5t+4\cos h4t+\cos 3t-\cos h5t$$

7. Find $L^{-1}\left(\frac{1}{2s+3}\right)$

Solution :

$$L^{-1}\left(\frac{1}{2s+3}\right)=\frac{1}{2}L^{-1}\left(\frac{1}{s+3/2}\right)$$
$$=\frac{1}{2}e^{-3/2t}$$

19. First Shiffting Property

(i) If $L^{-1}(F(s))=f(t)$ then $L^{-1}(F(s-a))=e^{at}L^{-1}(F(s))$

Proof :

We know that $L(f(t))=F(s)$ then $L(e^{at}f(t))=F(s-a)$

Hence $e^{at}f(t)=L^{-1}(F(s-a))$

$e^{at}L^{-1}(F(s))=L^{-1}(F(s-a))$

(ii) If $L^{-1}(F(s)) = f(t)$ Then $L^{-1}(F(s+a)) = e^{-at}L^{-1}(F(s))$

Proof :

We know that $L(f(t)) = F(s)$ Then $L(e^{-at}f(t)) = F(s+a)$

$$e^{-at}f(t) = L^{-1}(F(s+a))$$

Hence

$$e^{-at}L^{-1}(F(s)) = L^{-1}(F(s+a))$$

19.1. Problems :

1. Find $L^{-1}\left(\dfrac{1}{(s+1)^2}\right)$

Solution :

$$L^{-1}\left(\frac{1}{(s+1)^2}\right) = e^{-t}L^{-1}\left(\frac{1}{s^2}\right)$$
$$= e^{-t}t$$

2. Find $L^{-1}\left(\dfrac{1}{(s+1)^2+1}\right)$

Solution :

$$L^{-1}\left(\frac{1}{(s+1)^2+1}\right) = e^{-t}L^{-1}\left(\frac{1}{s^2+1}\right)$$
$$= e^{-t}\sin t$$

3. Find $L^{-1}\left(\dfrac{s-3}{(s-3)^2+4}\right)$

Solution :

$$L^{-1}\left(\frac{s-3}{(s-3)^2+4}\right) = e^{3t}L^{-1}\left(\frac{s}{s^2+4}\right)$$
$$= e^{3t}\cos 2t$$

4. Find $L^{-1}\left(\dfrac{s}{(s+2)^2}\right)$

Solution :

$$L^{-1}\left(\frac{S}{(s+2)^2}\right) = L^{-1}\left(\frac{s+2-2}{(s+2)^2}\right)$$
$$= L^{-1}\left(\frac{s+2}{(s+2)^2} - \frac{2}{(s+2)^2}\right)$$
$$= L^{-1}\left(\frac{1}{(s+2)}\right) - 2L^{-1}\left(\frac{1}{(s+2)^2}\right)$$
$$= e^{-2t} - 2e^{-2t}\cdot t$$
$$= e^{-2t}(1-2t)$$

5. Find $L^{-1}\left(\frac{s}{(s-1)^2+3}+\frac{3s}{(s+2)^2-5}\right)$

Solution :

$$L^{-1}\left(\frac{s}{(s-1)^2+3}+\frac{3s}{(s+2)^2-5}\right)=L^{-1}\left(\frac{s}{(s-1)^2+3}\right)+3L^{-1}\left(\frac{s}{(s+2)^2-5}\right)$$

$$=L^{-1}\left(\frac{s-1+1}{(s-1)^2+3}\right)+3L^{-1}\left(\frac{s+2-2}{(s+2)^2-5}\right)$$

$$=L^{-1}\left(\frac{s-1}{(s-1)^2+3}\right)+L^{-1}\left(\frac{1}{(s-1)^2+3}\right)$$

$$+3L^{-1}\left(\frac{s+2}{(s+2)^2-5}\right)-6L^{-1}\left(\frac{1}{(s+2)^2-5}\right)$$

$$=e^tL^{-1}\left(\frac{s}{s^2+3}\right)+e^tL^{-1}\left(\frac{1}{s^2+3}\right)+3e^{-2t}L^{-1}\left(\frac{s}{s^2-5}\right)$$

$$-6e^{-2t}L^{-1}\left(\frac{1}{s^2-5}\right)$$

$$=e^tL^{-1}\left(\frac{s}{s^2+\sqrt{3}^2}\right)+\frac{e^t}{\sqrt{3}}L^{-1}\left(\frac{\sqrt{3}}{s^2+\sqrt{3}^2}\right)$$

$$+3e^{-2t}L^{-1}\left(\frac{s}{s^2-\sqrt{5}^2}\right)-\frac{6}{\sqrt{5}}e^{-2t}L^{-1}\left(\frac{\sqrt{5}}{s^2-\sqrt{5}^2}\right)$$

$$=e^t\cos\sqrt{3}t+\frac{e^t}{\sqrt{3}}\sin\sqrt{3}t+3e^{-2t}\cos h\sqrt{5}t$$

$$-\frac{6}{\sqrt{5}}e^{-2t}\sin h\sqrt{5}t$$

6. Find $L^{-1}\left(\frac{3s-4}{s^2-8s+65}\right)$

Solution :

$$L^{-1}\left(\frac{3s-4}{s^2-8S+65}\right)=L^{-1}\left(\frac{3s-4}{(s-4)^2+49}\right)$$

$$=L^{-1}\left(\frac{3\left(s-4/3\right)}{(s-4)^2+49}\right)=3L^{-1}\left(\frac{s-4+4-4/3}{(s-4)^2+49}\right)$$

$$=3L^{-1}\left(\frac{s-4+8/3}{(s-4)^2+49}\right)$$

$$=3L^{-1}\left(\frac{s-4}{(s-4)^2+49}\right)+3\cdot 8/3\,L^{-1}\left(\frac{1}{(s-4)^2+49}\right)$$

$$=3e^{4t}L^{-1}\left(\frac{s}{s^2+49}\right)+8e^{4t}L^{-1}\left(\frac{1}{s^2+49}\right)$$

$$=3e^{4t}\cos 7t+\frac{8}{7}e^{4t}L^{-1}\left(\frac{7}{s^2+49}\right)$$

$$=3e^{4t}\cos 7t+\frac{8}{7}e^{4t}\sin 7t$$

20. Change of Scale Property

If $L(f(t)) = F(s)$, then $L^{-1}(F(as)) = \frac{1}{a} f\left(\frac{t}{a}\right),\ a > 0$

Proof :

$$F(s) = L(f(t))$$

$$= \int_0^\infty e^{-St} f(t)dt$$

$$F(as) = \int_0^\infty e^{-ast} f(t)dt$$

Let $at = t_1$ When $t = 0,\quad t_1 = 0$

$dt = \frac{dt_1}{a}$ $t = \infty,\quad t_1 = \infty$

$$F(as) = \int_0^\infty e^{-St_1} f\left(\frac{t_1}{a}\right)\frac{dt_1}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-St_1} f\left(\frac{t_1}{a}\right)dt_1$$

$$= \frac{1}{a}\int_0^\infty e^{-St} f\left(\frac{t}{a}\right)dt \qquad \left(\because \int_a^b f(t)dt = \int_a^b f(t_1)dt_1\right)$$

$$= \frac{1}{a}L\left(f\left(t/a\right)\right)$$

$$\therefore L^{-1}(F(as)) = \frac{1}{a} f\left(\frac{t}{a}\right)$$

20.1. Problems :

1. If $L^{-1}\left(\frac{s^2-1}{(s^2+1)^2}\right) = t\cos t$, then find $L^{-1}\left(\frac{9s^2-1}{(9s^2+1)^2}\right)$

Solution :

$$L^{-1}\left(\frac{s^2-1}{(s^2+1)^2}\right) = t\cos t$$

writing as for S,

$$L^{-1}\left(\frac{a^2s^2-1}{(a^2s^2+1)^2}\right) = \frac{1}{a}\cdot\frac{t}{a}\cos\left(\frac{t}{a}\right)$$

Put $a = 3,\ L^{-1}\left(\frac{9s^2-1}{(9s^2+1)^2}\right) = \frac{1}{3}\cdot\frac{t}{3}\cos\left(\frac{t}{3}\right)$

$$= \frac{t}{9}\cos\left(\frac{t}{3}\right)$$

2. Find $L^{-1}\left(\frac{s}{(2s^2-8)}\right)$

Solution :

We know that $L^{-1}\left(\frac{s}{(s^2-4^2)}\right)=\cos h4t$

Putting as for S,

$$L^{-1}\left(\frac{2s}{(2s)^2-4^2}\right)=\frac{1}{2}\cos h\left(\frac{4t}{2}\right)$$

$$L^{-1}\left(\frac{2s}{4s^2-16}\right)=\frac{1}{2}\cos h2t$$

(ie)

$$L^{-1}\left(\frac{s}{2s^2-8}\right)=\frac{1}{2}\cos h2t$$

3. Find $L^{-1}\left(\frac{s}{s^2a^2+b^2}\right)$

Solution :

$$\frac{s}{s^2a^2+b^2}=\frac{1}{a}\frac{as}{s^2a^2+b^2}$$

$$=\frac{1}{a}F(as) \quad \text{where} \quad F(as)=\frac{s}{s^2+b^2}$$

$$\therefore L^{-1}\left(\frac{s}{s^2a^2+b^2}\right)=\frac{1}{a}L^{-1}\left(\frac{sa}{s^2a^2+b^2}\right)$$

$$=\frac{1}{a}L^{-1}(F(as))$$

$$=\frac{1}{a}\cdot\frac{1}{a}f\left(\frac{t}{a}\right)$$

$$\text{where } f(t)=L^{-1}(F(s))=L^{-1}\left(\frac{s}{s^2+b^2}\right)=\cos bt$$

$$\therefore f\left(\frac{t}{a}\right)=\cos\left(\frac{bt}{a}\right)$$

$$\therefore L^{-1}\left(\frac{s}{s^2+b^2}\right)=\frac{1}{a}\cdot\frac{1}{a}\cos\left(\frac{bt}{a}\right)$$

$$=\frac{1}{a^2}\cos\left(\frac{bt}{a}\right)$$

21. Result :

We know that if $L(f(t))=F(S)$, then $L(tf(t))=\frac{-d}{ds}F(s)$

$$L(tf(t))=-F'(s)$$

Hence $L^{-1}(F'(s))=-tf(t)$

$$=-tL^{-1}(F(s))$$

$$\therefore L^{-1}(F'(s))=-tL^{-1}(F(s))$$

21.1. Problems :

1. Find $L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)$

Solution :

Let $F'(s)=\frac{s}{(s^2+a^2)^2}$

$$\frac{d}{ds}F(s)=\frac{s}{(s^2+a^2)^2}$$

$$\therefore F(s)=\int\frac{s}{(s^2+a^2)^2}ds$$

Put $s^2+a^2=u$

$$2sds=du$$

$$\therefore \int\frac{s}{(s^2+a^2)^2}ds=\int\frac{\frac{du}{2}}{u^2}$$

$$=\frac{-1}{2u}=\frac{-1}{2(s^2+a^2)}$$

$$\therefore F(s)=\frac{-1}{2(s^2+a^2)}$$

We know that $L(F'(s))=-tL^{-1}(F(s))$

$$\therefore L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)=-tL^{-1}\left(\frac{-1}{2(s^2+a^2)}\right)$$

$$=\frac{t}{2}L^{-1}\left(\frac{1}{s^2+a^2}\right)$$

$$=\frac{t}{2}\frac{1}{a}L^{-1}\left(\frac{a}{s^2+a^2}\right)$$

$$=\frac{t}{2a}\sin at$$

2. Find $L^{-1}\left(\frac{s+3}{(s^2+6s+13)^2}\right)$

Solution :

Let $\left(\frac{s+3}{(s^2+6s+13)^2}\right)=F'(s)$

$$\frac{dF(s)}{ds}=\frac{s+3}{(s^2+6s+13)^2}$$

$$\therefore F(s)=\frac{(s+3)ds}{(s^2+6s+13)^2}$$

Put $s^2+6s+13=u$

$(2s+6)ds=du$

$2(s+3)ds=du$

(ie) $F(s)=\int \frac{\frac{du}{2}}{u^2}=\frac{-1}{2u}$

$=\frac{-1}{2(s^2+6s+13)}$

We know that $L^{-1}(F'(s))=-tL^{-1}(F(s))$

$$\therefore L^{-1}\left(\frac{s+3}{(s^2+6s+13)^2}\right)=-tL^{-1}\left(\frac{-1}{2(s^2+6s+13)}\right)$$

$$=\frac{t}{2}L^{-1}\left(\frac{-1}{(s^2+6s+13)}\right)$$

$$=\frac{t}{2}L^{-1}\left(\frac{1}{(s+3)^2+2^2}\right)$$

$$=\frac{t}{2}e^{-3t}L^{-1}\left(\frac{1}{s^2+2^2}\right)$$

$$=\frac{t}{2}e^{-3t}\frac{1}{2}L^{-1}\left(\frac{2}{s^2+2^2}\right)$$

$$=\frac{t}{4}e^{-3t}\sin 2t$$

3. Find $L^{-1}\left(\frac{2(s+1)}{(s^2+2s+2)^2}\right)$

Solution :

$$F'(s)=\frac{2(s+1)}{(s^2+2s+2)^2}$$

$$\frac{dF(s)}{ds}=\frac{2(s+1)}{(s^2+2s+2)^2}$$

$$F(s)=\int\frac{2(s+1)}{(s^2+2s+2)^2}ds$$

Put $s^2+2s+2=u$

$(2s+2)ds=du$

$2(s+2)ds=du$

$$\therefore F(s) = \int \frac{du}{u^2}$$

$$= \frac{-1}{u}$$

$$= \frac{-1}{s^2 + 2s + 2}$$

$$\therefore L^{-1}\left(\frac{2(s+1)}{(s^2+2s+2)^2}\right) = -tL^{-1}\left(\frac{-1}{s^2+2s+2}\right)$$

$$= tL^{-1}\left(\frac{1}{s^2+2s+2}\right) = tL^{-1}\left(\frac{1}{(s+1)^2+1}\right)$$

$$= te^{-t}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= te^{-t}\sin t$$

4. Find $L^{-1}\left(\frac{s+2}{(s^2+4s+5)^2}\right)$

Solution :

Let $F'(s) = \frac{s+2}{(s^2+4s+5)^2}$

Integrate both sides w.r.t 'S'

$$F'(s) = \frac{s+2}{(s^2+4s+5)^2}$$

$$\int F'(s) = \int \frac{(s+2)ds}{(s^2+4s+5)^2}$$

$$F(s) = \int \frac{(s+2)ds}{(s^2+4s+5)^2}$$

Let $y = s^2 + 4s + 5$

$dy = (2s+4)ds$

$\frac{dy}{2} = (s+2)ds$

$$F(s) = \int \frac{dy/2}{y^2}$$

$$= \frac{1}{2}\int \frac{dy}{y^2}$$

$$= \frac{1}{2}\int y^{-2}dy$$

$$F(s) = \frac{1}{2}\left(\frac{y^{-2+1}}{-2+1}\right)$$

$$= \frac{-1}{2y}$$

$$= \frac{-1}{2(s^2+4s+5)}$$

We know that

$$L^{-1}(F'(s)) = -tL^{-1}(F(s))$$

$$L^{-1}\left(\frac{s+2}{(s^2+4s+5)^2}\right) = -tL^{-1}\left(\frac{-1}{2(s^2+4s+5)}\right)$$

$$L^{-1}\left(\frac{s+2}{(s^2+4s+5)^2}\right) = \frac{t}{2}L^{-1}\left(\frac{1}{s^2+4s+5}\right)$$

$$= \frac{t}{2}L^{-1}\left(\frac{1}{(s+2)^2+1}\right)$$

$$= \frac{t}{2}e^{-2t}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{t}{2}e^{-2t}\sin t$$

5. Find $L^{-1}\left(\tan^{-1}\left(1/s\right)\right)$

Solution :

Let $F(s) = \tan^{-1}\left(1/s\right)s$

$$F'(s) = \frac{1}{1+\left(1/s\right)^2}\left(\frac{-1}{s^2}\right) \qquad \left[\because \frac{d(\tan^{-1}x)}{dx} = \frac{1}{1+x^2}\right]$$

$$F'(s) = \frac{s^2}{s^2+1}\left(\frac{-1}{s^2}\right)$$

$$= \frac{-1}{s^2+1}$$

We know that $L^{-1}(F'(s)) = -tL^{-1}(F(s))$

Or

$$L^{-1}(F(s)) = \frac{-1}{t}L^{-1}(F'(s)) \qquad \text{\_\_\_\_\_\_(1)}$$

$\therefore$ (1) becomes, $L^{-1}\left(\tan^{-1}\left(1/s\right)\right) = \frac{-1}{t}L^{-1}F(s)$

$$= \frac{1}{t}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$L^{-1}\left(\tan^{-1}\left(1/s\right)\right) = \frac{1}{t}\sin t$$

6. Find $L^{-1}\left(\tan^{-1}\left(a/s\right)+\cot^{-1}\left(s/b\right)\right)$

Solution :

Let $F(s)=\tan^{-1}\left(a/S\right)s+\cot^{-1}\left(s/b\right)$

$$F'(s)=\frac{1}{1+\left(a/s\right)^2}\left(\frac{-a}{s^2}\right)+\frac{-1}{1+\left(s/b\right)^2}\left(\frac{1}{b}\right)$$

$$F'(s)=\frac{s^2}{s^2+a^2}\left(\frac{-a}{s^2}\right)-\frac{b^2}{b^2+s^2}\left(\frac{1}{b}\right)$$

$$F'(s)=\frac{-a}{s^2+a^2}-\frac{b}{b^2+s^2}$$

We know that $L^{-1}(F(s))=\frac{-1}{t}L^{-1}(F'(s))$

$$L^{-1}\left(\tan^{-1}\left(a/s\right)+\cot^{-1}\left(s/b\right)\right)=\frac{-1}{t}L^{-1}\left(\frac{-a}{s^2+a^2}-\frac{b}{b^2+s^2}\right)$$

$$=\frac{1}{t}L^{-1}\left(\frac{a}{s^2+a^2}-\frac{b}{b^2+s^2}\right)$$

$$=\frac{1}{t}L^{-1}\left(L^{-1}\left(\frac{a}{s^2+a^2}\right)-L^{-1}\left(\frac{b}{b^2+s^2}\right)\right)$$

$$=\frac{1}{t}(\sin at+\sin bt)$$

7. Find $L^{-1}\left(\log\left(1+a^2/s^2\right)\right)$

Solution :

Let $F(s)=\log\left(1+a^2/s^2\right)$

$$\therefore F(s)=\log\left(\frac{s^2+a^2}{s^2}\right)$$

$$F(s)=\log(s^2+a^2)-\log s^2$$

$$F(s)=\log(s^2+a^2)-2\log s$$

$$\therefore F'(s)=\frac{2s}{s^2+a^2}-\frac{2}{s}$$

We know that $L^{-1}(F(s))=\frac{-1}{t}L^{-1}(F'(s))$

$$L^{-1}\left(\log\left(1+a^2/s^2\right)\right)=\frac{-1}{t}L^{-1}\left(\frac{2s}{s^2+a^2}-\frac{2}{s}\right)$$

$$=\frac{-2}{t}\left(L^{-1}\left(\frac{s}{s^2+a^2}\right)-L^{-1}\left(\frac{1}{s}\right)\right)$$

$$=\frac{-2}{t}\left(\cos at-1\right)$$

$$=\frac{2}{t}\left(1-\cos at\right)$$

8. Find $L^{-1}\left(\log\frac{(s+a)}{(s+b)}\right)$

Solution :

Let $F(s)=\log\frac{(s+a)}{(s+b)}$

$$=\log(s+a)-\log(s+b)$$

$$F'(s)=\frac{1}{s+a}-\frac{1}{S+b} \qquad \because L^{-1}(F(s))=\frac{-1}{t}L^{-1}(F'(s))$$

$$L^{-1}\left(\log\frac{(s+a)}{(s+b)}\right)=\frac{-1}{t}L^{-1}\left(\frac{1}{s+a}-\frac{1}{s+b}\right)$$

$$=\frac{-1}{t}(e^{-at}-e^{-bt})$$

9. Find $L^{-1}\left(\log\frac{s(s^2+a^2)}{(s^2+b^2)}\right)$

Solution :

Let $F(s)=\log\frac{(s^2+a^2)}{(s^2+b^2)}$

$$F(s)=\log\left(s(s^2+a^2)-\log(s^2+b^2)\right)$$

$$F(s)=\log s+\log(s^2+a^2)-\log(s^2+b^2)$$

$$F'(s)=\frac{1}{s}+\frac{2s}{(s^2+a^2)}-\frac{2s}{(s^2+b^2)}$$

We know that $L^{-1}(F(s))=\frac{-1}{t}L^{-1}(F'(s))$

$$L^{-1}\left(\log\frac{s(s^2+a^2)}{s(s^2+b^2)}\right)=\frac{-1}{t}L^{-1}\left(\frac{1}{s}+\frac{2s}{s^2+a^2}-\frac{2s}{s^2+b^2}\right)$$

$$=\frac{-1}{t}\left(L^{-1}\left(\frac{1}{s}\right)+L^{-1}\left(\frac{2s}{s^2+a^2}\right)-L^{-1}\left(\frac{2s}{s^2+b^2}\right)\right)$$

$$=\frac{-1}{t}\left[1+2\cos at-2\cos bt\right]$$

10. Find $L^{-1}\left(\log\frac{s(s^2+1)(s-4)^2}{(s^2-9)(s^2+4)}\right)$

Solution :

Let $F(s)=\log\left(\frac{s(s^2+1)(s-4)^2}{(s^2-9)(s^2+4)}\right)$

$$=\log(s(s^2+1)(s-4)^2)-\log\left((s^2-9)(s^2+4)\right)$$

$$F(s)=\log s+\log(s^2+1)+\log(s-4)^2-\log(s^2-9)-\log(s^2+4)$$

$$F'(s)=\frac{1}{s}+\frac{2s}{s^2+1}+\frac{2(s-4)}{(s-4)^2}-\frac{2s}{s^2-9}-\frac{2s}{s^2+4}$$

we know that, $L^{-1}(F(s))=\frac{-1}{t}L^{-1}(F'(s))$

$$L^{-1}\left(\log\frac{s(s^2+1)(s-4)^2}{(s^2-9)(s^2+4)}\right)=\frac{-1}{t}L^{-1}\left(\frac{1}{s}+\frac{2s}{s^2+1}+\frac{2}{s-4}-\frac{2s}{s^2-9}-\frac{2s}{s^2+4}\right)$$

$$=\frac{-1}{t}\left(1+2\cos t+2e^{4t}-2\cos h3t-2\cos 2t\right)$$

11. Find $L^{-1}\left(\log\frac{s-a}{(s^2+a^2)}\right)$

Solution :

Let $F(s)=\log\frac{s-a}{s^2+a^2}$

$$=\log(s-a)-\log(s^2+a^2)$$

$$F'(s)=\frac{1}{s-a}-\frac{2s}{s^2+a^2}$$

We know that $L^{-1}(F(s))=\frac{-1}{t}L^{-1}(F'(s))$

$$L^{-1}\left(\log\frac{s-a}{(s^2+a^2)}\right)=\frac{-1}{t}L^{-1}\left(\frac{1}{s-a}-\frac{2s}{s^2+a^2}\right)$$

$$=\frac{-1}{t}L^{-1}\left(\frac{2s}{s^2+a^2}-\frac{1}{s-a}\right)$$

$$=\frac{1}{t}\left(L^{-1}\left(\frac{2s}{s^2+a^2}\right)-L^{-1}\left(\frac{1}{s-a}\right)\right)$$

$$=\frac{1}{t}(2\cos at-e^{at})$$

22. Theorem :

If $L(f(t)) = F(s)$ and $\varphi(t)$ is a function such that $L(\varphi(t)) = F(s)$ and $\varphi(0) = 0$, then $f(t) = \varphi'(t)$,

(ie), $L^{-1}(sf(s)) = \frac{d}{dt} L^{-1}(F(s))$.

Proof :

We know that

$$\begin{aligned} L(\varphi'(t)) &= sL(\varphi(t)) - \varphi(0) \\ &= sF(s) \qquad\qquad (\because \varphi(0) = 0) \\ (ie) L(\varphi'(t)) &= L(f(t)) \\ \therefore \varphi'(t) &= f(t) \end{aligned}$$

From this result, we get

$$\begin{aligned} L^{-1}(s(s)) &= f(t) \\ &= \varphi'(t) \\ &= \frac{d}{dt}\varphi(t) \\ &= \frac{d}{dt} L^{-1}(F(s)) \qquad (\because L(\varphi(t)) = F(s)) \end{aligned}$$

Provided $L^{-1}(F(s)) = 0$ as $t \to 0$

Problems :

1. Find $L^{-1}\left(\frac{s}{(s+2)^2+4}\right)$

Solution :

$$\begin{aligned} L^{-1}\left(\frac{s}{(s+2)^2+4}\right) &= L^{-1}\left(s \cdot \frac{1}{(s+2)^2+4}\right) \\ &= \frac{d}{dt}\left(\frac{1}{(s+2)^2+4}\right) \qquad \text{(using the above result)} \\ &= \frac{d}{dt} e^{-2t} L^{-1}\left(\frac{1}{s^2+4}\right) \\ &= \frac{d}{dt} e^{-2t} L^{-1}\left(\frac{1}{s^2+4}\right) \\ &= \frac{d}{dt}\left(e^{-2t}\frac{1}{2}\sin 2t\right) \\ &= \frac{1}{2}\left(2e^{-2t}\cos 2t + \sin 2t e^{-2t}(-2)\right) \\ &= e^{-2t}(\cos 2t - \sin 2t) \end{aligned}$$

Aliter :

$$L^{-1}\left(\frac{s}{(s+2)^2+4}\right)=L^{-1}\left(\frac{s+2-2}{(s+2)^2+4}\right)$$

$$=L^{-1}\left(\frac{s+2}{(s+2)^2+4}-\frac{2}{(s+2)^2+4}\right)$$

$$=L^{-1}\left(\frac{s+2}{(s+2)^2+4}\right)-2L^{-1}\left(\frac{1}{(s+2)^2+4}\right)$$

$$=e^{-2t}L^{-1}\left(\frac{s}{s^2+2^2}\right)-2e^{-2t}L^{-1}\left(\frac{1}{s^2+2^2}\right)$$

$$=e^{-2t}\cos 2t-2e^{-2t}\frac{1}{2}\sin 2t$$

$$=e^{-2t}(\cos 2t-\sin 2t)$$

2. Find $L^{-1}\left(\frac{s}{(s+2)^2}\right)$

Solution :

$$L^{-1}\left(\frac{s}{(s+2)^2}\right)=L^{-1}\left(\frac{s}{(s+2)^2}\right)$$

$$=L^{-1}\left(s\cdot\frac{1}{(s+2)^2}\right)$$

$$=\frac{d}{dt}L^{-1}\left(\frac{1}{(s+2)^2}\right)$$

$$=\frac{d}{dt}e^{-2t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$=e^{-2t}+t(e^{-2t}(-2))$$

$$=e^{-2t}(1-2t)$$

Aliter :

$$L^{-1}\left(\frac{s}{(s+2)^2}\right)=L^{-1}\left(\frac{s+2-2}{(s+2)^2}\right)$$

$$=L^{-1}\left(\frac{s+2}{(s+2)^2}\right)-L^{-1}\left(\frac{2}{(s+2)^2}\right)$$

$$=L^{-1}\left(\frac{1}{(s+2)}\right)-2e^{-2t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$=e^{-2t}-2e^{-2t}t$$

$$=e^{-2t}(1-2t)$$

3. Find $L^{-1}\left(\frac{s^2}{(s^2+a^2)^2}\right)$

Solution :

$$L^{-1}\left(\frac{s^2}{(s^2+a^2)^2}\right)=L^{-1}\left(s\cdot\frac{s}{(s^2+a^2)}\right)$$

$$=\frac{d}{dt}L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)$$

$$=\frac{d}{dt}\left(\frac{t}{2a}\sin at\right)$$ (By the Previous Section 21.1 Problem No.1)

$$=\frac{1}{2a}(at\cos at+\sin at)$$

4. Find $L^{-1}\left(\frac{s^2}{(s-1)^4}\right)$

Solution :

$$L^{-1}\left(\frac{s^2}{(s-1)^4}\right)=L^{-1}\left(s\cdot\frac{s}{(s-1)^4}\right)$$

$$=\frac{d}{dt}L^{-1}\left(\frac{s}{(s-1)^4}\right)$$

$$=\frac{d}{dt}L^{-1}\left(\frac{s-1+1}{(s-1)^4}\right)$$

$$=\frac{d}{dt}\left(L^{-1}\left(\frac{s-1}{(s-1)^4}\right)+L^{-1}\left(\frac{1}{(s-1)^4}\right)\right)$$

$$=\frac{d}{dt}\left(L^{-1}\left(\frac{1}{(s-1)^3}\right)+L^{-1}\left(\frac{1}{(s-1)^4}\right)\right)$$

$$=\frac{d}{dt}\left(e^tL^{-1}\left(\frac{1}{S^3}\right)+e^tL^{-1}\left(\frac{1}{S^4}\right)\right)$$

$$=\frac{d}{dt}\left(e^t\frac{t^2}{2}+e^t\frac{t^3}{6}\right)$$

$$=\frac{1}{2}\left(e^t2t+t^2e^t\right)+\frac{1}{6}\left(e^t3t^2+t^3e^t\right)$$

$$=te^t+e^tt^2+\frac{t^3e^t}{6}$$

5. Find $L^{-1}\left(\frac{s-3}{s^2+4s+13}\right)$

Solution :

$$L^{-1}\left(\frac{s-3}{s^2+4s+13}\right)=L^{-1}\left(\frac{s}{s^2+4s+13}\right)-L^{-1}\left(\frac{3}{s^2+4s+13}\right)$$
$$=\frac{d}{dt}L^{-1}\left(\frac{1}{s^2+4s+13}\right)-3L^{-1}\left(\frac{1}{s^2+4s+13}\right)$$
$$=\frac{d}{dt}L^{-1}\left(\frac{1}{(s+2)^2+9}\right)-3L^{-1}\left(\frac{1}{(s+2)^2+3^2}\right)$$
$$=\frac{d}{dt}e^{-2t}L^{-1}\left(\frac{1}{s^2+3^2}\right)-3e^{-2t}L^{-1}\left(\frac{1}{s^2+3^2}\right)$$
$$=\frac{d}{dt}\left(e^{-2t}\frac{\sin 3t}{3}\right)-3e^{-2t}\left(\frac{\sin 3t}{3}\right)$$
$$=\frac{1}{3}(3e^{-2t}\cos 3t-2\sin 3te^{-2t})-e^{-2t}\sin 3t$$
$$=e^{-2t}\cos 3t-\frac{5}{3}e^{-2t}\sin 3t$$

23. Theorem :

$$L^{-1}\left(\frac{F(s)}{s}\right)=\int_0^t L^{-1}(F(s))dt$$

Proof :

We know that,

$$L\left(\int_0^t f(x)dx\right)=\frac{1}{s}L(f(t))$$

$$\therefore \int_0^t f(x)dx=L^{-1}\left(\frac{1}{s}L(f(t))\right)$$

$$(ie)L^{-1}\left(\frac{1}{s}F(s)\right)=\int_0^t f(t)dt \qquad \text{s}\left[\because F(s)=L(f(t))\right]$$
$$=\int_0^t L^{-1}(F(s))dt$$

$$\therefore L^{-1}\left(\frac{1}{s}F(s)\right)=\int_0^t L^{-1}(F(s))dt$$

Note :

Similarly $$L^{-1}\left(\frac{1}{s^2}F(s)\right)=\int_0^t\int_0^t L^{-1}(F(s))dtdt$$
$$L^{-1}\left(\frac{1}{s^3}F(s)\right)=\int_0^t\int_0^t\int_0^t L^{-1}(F(s))dtdtdt$$
$$L^{-1}\left(\frac{1}{s^n}F(s)\right)=\underbrace{\int_0^t\int_0^t\cdots\cdots\int_0^t}_{n\text{ times}} L^{-1}(F(s))\underbrace{dtdt\cdots\cdots dt}_{n\text{ times}}$$

23.1. Problems :

1. Find $L^{-1}\left(\frac{1}{s(s+1)}\right)$

Solution :

$$L^{-1}\left(\frac{1}{s(s+1)}\right)=\int_0^t L^{-1}\left(\frac{1}{(s+1)}\right)dt \qquad \text{(by the above theorem)}$$

$$=\int_0^t e^{-t}dt$$

$$=\left(-e^{-t}\right)_0^t$$

$$=-\left(e^{-t}-1\right)$$

$$=1-e^{-t}$$

2. Find $L^{-1}\left(\frac{1}{s(s+2)^3}\right)$

Solution :

$$L^{-1}\left(\frac{1}{s(s+2)^3}\right)=\int_0^t\left(\frac{1}{(s+2)^3}\right)dt$$

$$=\int_0^t e^{-2t}L^{-1}\left(\frac{1}{s^3}\right)dt$$

$$=\int_0^t \frac{e^{-2t}}{2}L^{-1}\left(\frac{2}{s^3}\right)dt$$

$$=\frac{1}{2}\int_0^t e^{-2t}t^2dt$$

$$=\frac{1}{2}\left[(t^2)\left(\frac{e^{-2t}}{-2}\right)-(2t)\left(\frac{e^{-2t}}{4}\right)+2\left(\frac{e^{-2t}}{-8}\right)\right]_0^t$$

$$\left[\because \int udv=uv-u'v_1+u''v_2\cdots\right]$$

$$=\frac{1}{2}\left[\frac{-t^2e^{-2t}}{2}-\frac{te^{-2t}}{2}-\frac{e^{-2t}}{4}+\frac{1}{4}\right]$$

$$=\frac{1}{2}\left[\frac{-e^{-2t}}{2}\left(t^2+t+\frac{1}{2}\right)+\frac{1}{4}\right]$$

$$=\frac{1}{8}\left(1-(2t^2+2t+1)e^{-2t}\right)$$

3. Find $L^{-1}\left(\frac{54}{s^3(s-3)}\right)$

Solution :

$$L^{-1}\left(\frac{54}{s^3(s-3)}\right)=54\int_0^t\int_0^t\int_0^t L^{-1}\left(\frac{1}{(s-3)}\right)dtdtdt$$

$$=54\int_0^t\int_0^t\int_0^t e^{3t}dtdtdt$$

$$=54\int_0^t\int_0^t\left(\frac{e^{3t}}{3}\right)_0^t dtdt$$

$$=18\int_0^t\int_0^t(e^{3t}-1)dtdt$$

$$=18\int_0^t\left(\frac{e^{3t}}{3}-t\right)_0^t dt$$

$$=18\int_0^t\left(\frac{e^{3t}}{3}-t\right)-\left(\frac{1}{3}-0\right)dt$$

$$=18\int_0^t\left(\frac{e^{3t}}{3}-t-\frac{1}{3}\right)dt$$

$$=18\left(\frac{e^{3t}}{9}-\frac{t^2}{2}-\frac{1}{3}t\right)_0^t$$

$$=18\left(\frac{e^{3t}}{9}-\frac{t^2}{2}-\frac{t}{3}-\frac{1}{9}\right)$$

$$=2e^{3t}-9t^2-6t-2$$

4. Find $L^{-1}\left(\frac{1}{s(s^2+a^2)}\right)$

Solution :

$$L^{-1}\left(\frac{1}{s(s^2+a^2)}\right)=\int_0^t L^{-1}\left(\frac{1}{s^2+a^2}\right)dt$$

$$=\int_0^t\frac{1}{a}L^{-1}\left(\frac{a}{s^2+a^2}\right)dt$$

$$=\frac{1}{a}\int_0^t\sin at dt$$

$$=\frac{1}{a}\left(\frac{-\cos at}{a}\right)_0^t$$

$$=\frac{-1}{a^2}(\cos at-1)$$

$$=\frac{+1}{a^2}(1-\cos at)$$

5. Find $L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right)$

Solution :

$$L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right)=L^{-1}\left(\frac{s}{s(s^2+a^2)^2}\right)$$

$$=L^{-1}\left(\frac{1}{s}\cdot\frac{s}{(s^2+a^2)^2}\right)$$

$$=\int_0^t L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)dt$$

$$=\int_0^t \frac{t\sin at}{2a}\,dt$$

$$=\frac{1}{2a}\left(t\left(\frac{-\cos at}{a}\right)-1\left(\frac{-\sin at}{a^2}\right)\right)_0^t$$

$$=\frac{1}{2a}\left(\frac{-t\cos at}{a}+\frac{\sin at}{a^2}\right)$$

(By the previous section 21.1 Problem no.1)

6. Find $L^{-1}\left(\frac{1}{s(s^2-2s+5)}\right)$

Solution :

$$L^{-1}\left(\frac{1}{s(s^2-2s+5)}\right)=L^{-1}\left(\frac{1}{s}\cdot\frac{1}{s^2-2s+5}\right)$$

$$=\int_0^t L^{-1}\left(\frac{1}{s^2-2s+5}\right)dt$$

$$=\int_0^t L^{-1}\left(\frac{1}{(s-1)^2+2^2}\right)dt$$

$$=\int_0^t e^t L^{-1}\left(\frac{1}{s^2+2^2}\right)dt$$

$$=\int_0^t e^t\,\frac{\sin 2t}{2}\,dt$$

$$=\frac{1}{2}\int_0^t e^t\sin 2t\;dt$$

$$=\frac{1}{2}\left[\frac{e^t}{1^2+2^2}(\sin 2t-2\cos 2t)\right]_0^t$$

$$=\frac{1}{10}\left[e^t\sin 2t-2e^t\cos 2t\right]_0^t$$

$$=\frac{1}{10}\left[e^t\sin 2t-2e^t\cos 2t-0+2\right]$$

$$=\frac{1}{10}\left[e^t\sin 2t-2e^t\cos 2t+2\right]$$

7. Find $L^{-1}\left(\frac{1}{s(s^2+6s+13)}\right)$

Solution :

$$L^{-1}\left(\frac{1}{s(s^2+6s+13)}\right)=L^{-1}\left(\frac{1}{s}\cdot\frac{1}{s^2+6s+13}\right)$$

$$=\int_0^t L^{-1}\left(\frac{1}{(s+3)^2+4}\right)dt$$

$$=\int_0^t e^{-3t}L^{-1}\left(\frac{1}{s^2+4}\right)dt$$

$$=\frac{1}{2}\int_0^t e^{-3t}L^{-1}\left(\frac{2}{s^2+4}\right)dt$$

$$=\frac{1}{2}\int_0^t e^{-3t}\sin 2t\ dt$$

$$=\frac{1}{2}\left\{\frac{e^{-3t}}{(-3)^2+2^2}(-3\sin 2t-2\cos 2t)\right\}_0^t$$

$$=\frac{-1}{26}\left\{e^{-3t}(3\sin 2t+2\cos 2t)-2\right\}$$

8. Find $L^{-1}\left(\frac{1}{s(s^2+a^2)^2}\right)$

Solution :

$$L^{-1}\left(\frac{1}{s(s^2+a^2)^2}\right)=L^{-1}\left(\frac{s}{s^2(s^2+a^2)^2}\right)$$

$$=\int_0^t\int_0^t L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right)dtdt$$

$$=\int_0^t\int_0^t \frac{t}{2a}\sin atdtdt$$ (refer the above problem)

$$=\frac{1}{2a}\int_0^t\int_0^t t\sin atdtdt$$

$$=\frac{1}{2a}\int_0^t\left(t\left(\frac{-\cos at}{a}\right)-(1)\left(\frac{-\sin at}{a^2}\right)\right)dt$$

$$=\frac{1}{2a}\int_0^t\left(\frac{\sin at}{a^2}-\frac{t\cos at}{a}\right)_0^t dt$$

$$=\frac{1}{2a^3}\int_0^t\left(\sin at-at\cos at\right)dt$$

$$=\frac{1}{2a^3}\left[\left(\frac{-\cos at}{a}\right)_0^t-a\left(t\left(\frac{\sin at}{a}\right)-(1)\left(\frac{-\cos at}{a^2}\right)\right)_0^t\right]$$

$$= \frac{1}{2a^3}\left[\frac{-\cos at}{a} - t\sin at - \frac{-\cos at}{a}\right]_0^t$$

$$= \frac{-1}{2a^3}\left[\frac{2\cos at}{a} + t\sin at\right]_0^t$$

$$= \frac{-1}{2a^3}\left[\frac{2\cos at}{a} + t\sin at - \frac{2}{a}\right]$$

$$= \frac{1}{2a^4}(2 - 2\cos at - at\sin at)$$

Inverse Laplace Transform using Second Shifting Theorem

If $L(f(t)) = F(s)$, then $L(f(t-a)\cdot U(t-a)) = e^{-as}F(s)$ where 'a' is a positive constant and $U(t-a)$ is the unit step function.

The above property can be written in terms of inverse Laplace operator as,

If $L^{-1}(F(s)) = f(t)$ then $L^{-1}(e^{-as}F(s)) = f(t-a)U(t-a)$

$\therefore L^{-1}(e^{-as}F(s)) = L^{-1}(F(s))_{t\to t-a}\cdot U(t-a)$ where U is the unit step function.

Thus we want to find the Laplace inverse transform of the product of two factors one of which is e^{-as}, ignore e^{-as}, find the inverse transform of the other function and then replace *t* by $t-a$ in it and multiply by $U(t-a)$

Problems :

1. Find $L^{-1}\left(\frac{e^{-S}}{s+2}\right)$

Solution :

$$L^{-1}\left(\frac{e^{-S}}{s+2}\right) = L^{-1}\left(\frac{1}{s+2}\right)_{t\to t-1}\cdot U(t-1)$$

$= (e^{-2t})_{t\to t-1}\cdot U(t-1)$ where U is the unit step function.

$= e^{-2(t-1)}U(t-1)$

2. Find $L^{-1}\left(\frac{e^{-2S}}{s-1}\right)$

Solution :

$$L^{-1}\left(\frac{e^{-2S}}{s-1}\right) = \left\{L^{-1}\left(\frac{1}{s-1}\right)\right\}_{t\to t-2}\cdot U(t-2)$$

$= (e^{t})_{t\to t-2}U(t-2)$ where U is the unit step function

$= e^{t-2}U(t-2)$

3. Find $L^{-1}\left(\dfrac{e^{-S}}{(s+1)^{5/2}}\right)$

Solution :

$$L^{-1}\left(\frac{e^{-S}}{(s+1)^{5/2}}\right)=\left\{L^{-1}\left(\frac{1}{(s+1)^{5/2}}\right)\right\}_{t\to t-1}U(t-1) \quad \_\_\_\_\_\_\_(1)$$

Now, $L^{-1}\left(\dfrac{1}{(s+1)^{5/2}}\right)=e^{-t}L^{-1}\left(\dfrac{1}{s^{5/2}}\right)$ Using first shifting property.

$$=e^{-t}\frac{1}{\Gamma\left(5/2\right)}t^{3/2} \qquad \left(\because L^{-1}\left(\frac{1}{s^n}\right)=\frac{1}{\Gamma(n)}t^{n-1}\right)$$

$$=e^{-t}\frac{1}{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}t^{3/2}$$

$$=\frac{4}{3\sqrt{\pi}}e^{-t}t^{3/2} \quad \_\_\_\_\_\_\_(2)$$

Substituting (2) in (1)

$$L^{-1}\left(\frac{e^{-S}}{(s+1)^{5/2}}\right)=\left(\frac{4}{3\sqrt{\pi}}e^{-t}t^{3/2}\right)_{t\to t-1}U(t-1)$$

$$L^{-1}\left(\frac{e^{-S}}{(s+1)^{5/2}}\right)=\frac{4}{3\sqrt{\pi}}e^{-(t-1)}(t-1)^{3/2}\cdot U(t-1)$$

4. Find $L^{-1}\left(\dfrac{se^{-aS}}{s^2-w^2}\right),\ a>0$

Solution :

$$L^{-1}\left(\frac{se^{-aS}}{s^2-w^2}\right)=\left\{L^{-1}\left(\frac{s}{s^2-w^2}\right)\right\}_{t\to t-a}U(t-a)$$

$$=(\cosh wt)_{t\to t-a}\cdot U(t-a)$$

$$=\cosh wt(t-a)\cdot U(t-a)$$

5. Find $L^{-1}\left(\dfrac{e^{-2S}}{(s+1)^3}\right)$

Solution :

$$L^{-1}\left(\frac{e^{-2S}}{(s+1)^3}\right)=\left\{L^{-1}\left(\frac{1}{(s+1)^3}\right)\right\}_{t\to t-2}\cdot U(t-2) \quad \_\_\_\_\_\_(1)$$

Now, $L^{-1}\left(\dfrac{1}{(s+1)^3}\right)=e^{-t}L^{-1}\left(\dfrac{1}{s^3}\right)$

$$= \frac{e^{-t}}{2!} L^{-1}\left(\frac{2!}{S^3}\right)$$

$$= \frac{e^{-t}}{2} t^2 \qquad \text{\_\_\_\_\_\_(2)}$$

Substituting (2) in (1)

$$L^{-1}\left(\frac{e^{-2S}}{(s+1)^3}\right) = \left(\frac{e^{-t}}{2} t^2\right)_{t \to t-2} U(t-2)$$

$$= \frac{e^{-(t-2)} \cdot (t-2)^2 U(t-2)}{2}$$

6. Find $L^{-1}\left(\left(\frac{3a-4s}{s^2+a^2}\right)e^{-5s}\right)$

Solution:

$$L^{-1}\left(e^{-5s}\left(\frac{3a-4s}{s^2+a^2}\right)\right) = L^{-1}\left(\left(\frac{3a-4s}{s^2+a^2}\right)\right)_{t \to t-5} \cdot U(t-5)$$

$$= \left[3L^{-1}\left(\frac{a}{a^2+s^2}\right) - 4L^{-1}\left(\frac{s}{a^2+s^2}\right)\right]_{t \to t-5} \cdot U(t-5)$$

$$= (3\sin at - 4\cos at)_{t \to t-5} U(t-5)$$

$$= 3\sin a(t-5) - 4\cos a(t-5) \cdot U(t-5)$$

6. Find $L^{-1}\left(\frac{e^{-\pi s}}{(s-2)(S+5)}\right)$

Solution :

$$L^{-1}\left(\frac{e^{-\pi s}}{(s-2)(s+5)}\right) = L^{-1}\left(\frac{1}{(s-2)(s+5)}\right)_{t \to t-\pi}$$

Now, $\frac{1}{(s-2)(s+5)} = \frac{A}{s-2} + \frac{B}{s+5}$

$$1 = A(s+5) + B(s-2)$$

Put $s = -5$ Put $s = 2$

$\therefore B = \frac{-1}{7}$ $\therefore A = \frac{1}{7}$

$$\therefore L^{-1}\left(\frac{1}{(s-2)(s+5)}\right) = \frac{1}{7} L^{-1}\left(\frac{1}{s-2}\right) - \frac{1}{7} L^{-1}\left(\frac{1}{s+5}\right)$$

$$= \frac{1}{7} e^{2t} - \frac{1}{7} e^{-5t}$$

$$\therefore L^{-1}\left(\frac{e^{-\pi S}}{(s-2)(s+5)}\right) = \left(\frac{e^{2t}}{7} - \frac{e^{5t}}{7}\right)_{t \to t-\pi} U(t-\pi)$$

$$= \left(\frac{e^{2(t-\pi)}}{7} - \frac{e^{-5(t-\pi)}}{7}\right) U(t-\pi)$$

24. Partial Fraction

The rational fraction P(x)/Q(x) is said to be resolved into partial fraction if it can be expressed as the sum of difference of simple proper fractions.

Rules for resolving a Proper Fraction P(x) / Q(x) into partial fractions.

Rule 1 :

Corresponding to every non repeated, linear factor (ax+b) of the denomiator Q(x), there exists a partial fraction of the form $\frac{A}{ax+b}$ where A is a constant, to be determined.

For Example :

(i) $$\frac{2x-7}{(x-2)(3x-5)}=\frac{A}{x-2}+\frac{B}{3x-5}$$

(ii) $$\frac{5x^2+18x+22}{(x-1)(x+2)(2x+3)}=\frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{2x+3}$$

Rule 2 :

Corresponding to every repeated linear factor $(ax+b)^k$ of the denominator Q(x), there exist k partial fractions of the forms,

$$\frac{A_1}{ax+b},\frac{A_2}{(ax+b)^2},\frac{A_3}{(ax+b)^3},\cdots\cdots\frac{A_k}{(ax+b)^k}$$

where $A_1, A_2 \cdots\cdots A_k$ are constants to be detemined.

For example :

(i) $$\frac{4x-3}{(x+2)(2x-3)^2}=\frac{A}{x+2}+\frac{B}{2x-3}+\frac{C}{(2x-3)^2}$$

(ii) $$\frac{x+2}{(x-1)(2x+1)^3}=\frac{A}{x-1}+\frac{B}{(2x+1)}+\frac{C}{(2x+1)^2}+\frac{D}{(2x+1)^3}$$

Rule 3 :

Corresponding to every non-repeated irreducible quadratic factor ax^2+bx+c of the denominator Q(x) there exists a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$ where A and B are constants to be determined. (ax^2+bx+c) is said to be an irreducible quadratic factor, if it cannot be factorized into two linear fractors with real coefficients.

Example :

(i) $$\frac{x^2+1}{(x^2+4)(x^2+9)}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{x^2+9}$$

(ii) $$\frac{8x^3-5x^2+2x+4}{(2x-1)^2(3x^2+4)}=\frac{A}{2x-1}+\frac{B}{(2x-1)^2}+\frac{Cx+D}{3x^2+4}$$

In the case of an improper fraction, by division, it can be expressed as the sum of integral function and a proper fraction and then proper fraction is resolved into partial fractions.

Inverse Laplace Transform using Partial Fractions :

1. Find $L^{-1}\left(\dfrac{1}{(s+1)(s+3)}\right)$

Solution :

Let $F(s)=\left(\dfrac{1}{(s+1)(s+3)}\right)$

Let us split $F(S)$ into partial fractions,

$$\frac{1}{(s+1)(s+3)}=\frac{A}{(s+1)}+\frac{B}{(s+3)}$$

$$1=A(S+3)+B(S+1)$$

Putting $s=-1$ Putting $s=-3$

$A=1/2$ $B=-1/2$

$$\therefore \frac{1}{(s+1)(s+3)}=\frac{1/2}{(s+1)}+\frac{-1/2}{(s+3)}$$

$$\therefore \left(\frac{1}{(s+1)(s+3)}\right)=\frac{1}{2}L^{-1}\left(\frac{1}{s+1}\right)-\frac{1}{2}L^{-1}\left(\frac{1}{s+3}\right)$$

$$=\frac{1}{2}e^{-t}-\frac{1}{2}e^{-3t}$$

$$=\frac{1}{2}(e^{-t}-e^{-3t})$$

2. Find $L^{-1}\left(\dfrac{s^2+s-2}{s(s+3)(s-2)}\right)$

Solution :

Consider, $\dfrac{s^2+s-2}{s(s+3)(s-2)}=\dfrac{A}{s}+\dfrac{B}{s+3}+\dfrac{C}{s-2}$

$$\frac{s^2+s-2}{s(s+3)(s-2)}=\frac{A(s+3)(s-2)+Bs(s-2)+Cs(s+3)}{s(s+3)(s-2)}$$

$$s^2+s-2=A(s+3)(s-2)+Bs(s-2)+Cs(s+3)$$

put $s=-3$

$9-3-2=B(-3)(-5)$

$4=15B$

$B=\dfrac{4}{15}$

put $s=2$

$4+2-2=C(2)(5)$

$4=10C$

$\therefore C=\dfrac{4}{10}$

$C=\dfrac{2}{5}$

put $s=0$

$-2=A(3)(-2)$

$A=\dfrac{1}{3}$

$$\frac{s^2+s-2}{s(s+3)(s-2)}=\frac{1}{3}\cdot\frac{1}{s}+\frac{4}{15}\cdot\frac{1}{s+3}+\frac{2}{5}\cdot\frac{1}{s-2}$$

$$\therefore L^{-1}\left(\frac{s^2+s-2}{s(s+3)(s-2)}\right)=\frac{1}{3}L^{-1}\left(\frac{1}{s}\right)+\frac{4}{15}L^{-1}\left(\frac{1}{s+3}\right)+\frac{2}{5}L^{-1}\left(\frac{1}{s-2}\right)$$

$$=\frac{1}{3}(1)+\frac{4}{15}e^{-3t}+\frac{2}{5}e^{2t}$$

3. Find $L^{-1}\left(\dfrac{s}{s^2+5s+6}\right)$

Solution :

Consider $\dfrac{s}{s^2+5s+6}=\dfrac{s}{(s+2)(s+3)}=\dfrac{A}{(s+2)}+\dfrac{B}{(s+3)}$

$$s=A(s+3)+B(s+2)$$

Put $s=-3$

$-3=A(0)+B(-1)$

$-3=-B$

$B=3$

Put $s=-2$

$-2=A(1)+B(0)$

$A=-2$

$$\frac{s}{(s+2)(s+3)}=\frac{-2}{(s+2)}+\frac{3}{(s+3)}$$

$$\therefore L^{-1}\left(\frac{s}{(s+2)(s+3)}\right)=-2L^{-1}\left(\frac{1}{(s+2)}\right)+3L^{-1}\left(\frac{B}{(s+3)}\right)$$

$$=-2e^{-2t}+3e^{-3t}$$

4. Find $L^{-1}\left(\dfrac{s}{(s+1)^2}\right)$

Solution :

Consider $\dfrac{s}{(s+1)^2}=\dfrac{A}{s+1}+\dfrac{B}{(s+1)^2}$

$$\frac{S}{(s+1)^2}=\frac{A(s+1)+B}{(s+1)^2}$$

$$s=A(s+1)+B$$

Put $s=-1$

$B=-1$

Put $s=0$

$0=A+B$

$0=A-1$

$A=1$

$$\frac{s}{(s+1)^2} = \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

$$L^{-1}\left(\frac{s}{(s+1)^2}\right) = L^{-1}\left(\frac{1}{s+1} - \frac{1}{(s+1)^2}\right)$$

$$= L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{1}{(s+1)^2}\right)$$

$$= e^{-t} - e^{-t} L^{-1}\left(\frac{1}{s^2}\right)$$

$$= e^{-t} - e^{-t}(t) = e^{-t}(1-t)$$

5. Find $L^{-1}\left(\frac{5s^2 - 15s - 11}{(s+1)(s-2)^3}\right)$

Solution :

$$\frac{5s^2 - 15S - 11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{(s-2)} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)^3}$$

$$5s^2 - 15s - 11 = A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1)$$

Put $s = -1$

$$-27A = 9$$

$$A = \frac{-9}{27}$$

$$A = \frac{-1}{3}$$

Put $s = 2$

$$3D = -21$$

$$D = -7$$

Equating the coefficient of s^3

$$A + B = 0$$

$$B = -A$$

$$B = \frac{1}{3}$$

Equating the constant coefficient

$$-8A + 4B - 2C + D = -11$$

$$\frac{8}{3} + \frac{4}{3} - 2C - 7 = -11$$

$$-2C = -8$$

$$C = 4$$

$$\therefore \frac{5s^2 - 15s - 11}{(s+1)(s-2)^2} = \frac{\frac{-1}{3}}{s+1} + \frac{\frac{1}{3}}{s-2} + \frac{4}{(s-2)^2} - \frac{7}{(s-2)^3}$$

$$L^{-1}\left(\frac{5s^2 - 15s - 11}{(s+1)(s-2)^2}\right) = \frac{-1}{3} L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{3} L^{-1}\left(\frac{1}{s-2}\right)$$

$$+ 4L^{-1}\left(\frac{1}{(s-2)^2}\right) - 7L^{-1}\left(\frac{1}{(s-2)^3}\right)$$

$$= \frac{-1}{3} e^{-t} + \frac{1}{3} e^{2t} + 4e^{2t} L^{-1}\left(\frac{1}{s^2}\right) - 7e^{2t} L^{-1}\left(\frac{1}{s^3}\right)$$

$$= \frac{-1}{3} e^{-t} + \frac{1}{3} e^{2t} + 4e^{2t} \cdot t - \frac{7}{2} e^{2t} t^2$$

6. Find $L^{-1}\left(\dfrac{2s^2+5s+2}{(s-3)^4}\right)$

Solution :

To resolve $\dfrac{2s^2+5s+2}{(s-3)^4}$ into partial fraction

we substitute $s-3=y$ (or) $s=y+3$

$$\therefore \frac{2s^2+5S+2}{(s-3)^4}=\frac{2(y+3)^2+5(y+3)+2}{y^4}$$

$$=\frac{2(y^2+6y+9)+5y+15+2}{y^4}$$

$$=\frac{2y^2+17y+35}{y^4}$$

$$=\frac{2}{y^2}+\frac{17}{y^3}+\frac{35}{y^4}$$

$$\frac{2s^2+5s+2}{(s-3)^4}=\frac{2}{(s-3)^2}+\frac{17}{(s-3)^3}+\frac{35}{(s-3)^4}$$

$$\therefore L^{-1}\left(\frac{2s^2+5s+2}{(s-3)^4}\right)=2L^{-1}\left(\frac{1}{(s-3)^2}\right)+17L^{-1}\left(\frac{1}{(s-3)^3}\right)+35L^{-1}\left(\frac{1}{(s-3)^4}\right)$$

$$=2e^{3t}L^{-1}\left(\frac{1!}{s^2}\right)+\frac{17}{2}e^{3t}L^{-1}\left(\frac{2!}{s^3}\right)+\frac{35}{6}e^{3t}L^{-1}\left(\frac{3!}{s^4}\right)$$

$$=2e^{3t}\cdot t+\frac{17}{2}e^{3t}t^2+\frac{35}{6}t^3e^{3t}$$

7. Find $L^{-1}\left(\dfrac{s^2}{(s^2+a^2)(s+b^2)}\right)$

Solution :

$$\frac{s^2}{(s^2+a^2)(s^2+b^2)}=\frac{A}{(s^2+a^2)}+\frac{B}{(s^2+b^2)}$$

$$s^2=A(s^2+b^2)+B(s^2+a^2)$$

Put $s^2=-a^2,\quad -a^2=A(-a^2+b^2)$

$$A=\frac{-a^2}{b^2-a^2}=\frac{a^2}{a^2-b^2}$$

Put $s^2=-b^2,\quad -b^2=B(-b^2+a^2)$

$$B=\frac{-b^2}{a^2-b^2}$$

$$\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{\frac{a^2}{a^2-b^2}}{(s^2+a^2)} + \frac{\frac{-b^2}{a^2-b^2}}{(s^2+b^2)}$$

$$= \frac{1}{a^2-b^2}\left(\frac{a^2}{s^2+a^2} - \frac{b^2}{s^2+b^2}\right)$$

$$L^{-1}\left(\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right) = \frac{1}{a^2-b^2} L^{-1}\left(\frac{a^2}{s^2+a^2} - \frac{b^2}{s^2+b^2}\right)$$

$$= \frac{1}{a^2-b^2}\left(L^{-1}\left(\frac{a^2}{s^2+a^2}\right) - L^{-1}\left(\frac{b^2}{s^2+b^2}\right)\right)$$

$$= \frac{1}{a^2-b^2}(a\sin at - b\sin bt)$$

8. Find $L^{-1}\left(\frac{1-s}{(s+1)^2(s^2+4s+13)}\right)$

Solution :

$$\frac{1-s}{(s+1)(s^2+4s+13)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4s+13}$$

$$1-s = A(s^2+4s+13) + (Bs+C)(s+1)$$

Putting $s=-1$

$$2 = 10A$$

$$A = \frac{1}{5}$$

Equating coefficient of s^2

$$A+B=0$$

$$B=-A$$

$$B=\frac{-1}{5}$$

Equating constant coefficient

$$13A+C=1$$

$$13\left(\frac{1}{5}\right)+C=1$$

$$C = 1-\frac{13}{5}$$

$$C = \frac{-8}{5}$$

(ie), $\frac{1-s}{(s+1)(s^2+4s+13)} = \frac{\frac{1}{5}}{s+1} + \frac{\frac{-1}{5}s - \frac{8}{5}}{s^2+4s+13}$

$$L^{-1}\left(\frac{1-s}{(s+1)(s^2+4s+13)}\right) = \frac{1}{5}L^{-1}\left(\frac{1}{s+1}\right) - \frac{1}{5}L^{-1}\left(\frac{s+8}{s^2+4s+13}\right)$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}L^{-1}\left(\frac{s+2+6}{(s+2)^2+9}\right)$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}L^{-1}\left(\frac{s+2}{(s+2)^2+3^2}\right) - \frac{1}{5}L^{-1}\left(\frac{6}{(s+2)^2+3^2}\right)$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\cos 3t - \frac{6}{5}e^{-2t}\frac{\sin 3t}{3}$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\cos 3t - \frac{2}{5}e^{-2t}\sin 3t$$

9. Find $L^{-1}\left(\dfrac{4s^2-3s+5}{(s+1)(s^2-3s+2)}\right)$

Solution :

$$\frac{4s^2-3s+5}{(s+1)(s^2-3s+2)}=\frac{A}{s+1}+\frac{Bs+C}{s^2-3s+2}$$

$$4s^2-3s+5=A(s^2-3s+2)+(Bs+C)(s+1)$$

Putting $s=-1$

$$6A=12$$

$$A=2$$

Equating coefficient s^2

$$4=A+B$$

$$B=2$$

Equating constant coefficients

$$5=2A+C$$

$$C=5-2A$$

$$C=1$$

$$\therefore \frac{4s^2-3s+5}{(s+1)(s^2-3s+2)}=\frac{2}{s+1}+\frac{2s+1}{s^2-3s+2}$$

$$L^{-1}\left(\frac{4s^2-3s+5}{(s+1)(s^2-3s+2)}\right)=L^{-1}\left(\frac{2}{s+1}\right)L^{-1}\left(\frac{2s+1}{s^2-3s+2}\right)$$

$$=2L^{-1}\left(\frac{1}{s+1}\right)+L^{-1}\left(\frac{2s+1}{(s-3/2)^2-1/4}\right)$$

$$=2e^{-t}+2L^{-1}\left(\frac{s+1/2}{\left(s-3/2\right)^2-1/4}\right)$$

$$=2e^{-t}+2L^{-1}\left(\frac{s+1/2-2+2}{\left(s-3/2\right)^2-1/4}\right)$$

$$=2e^{-t}+2L^{-1}\left(\frac{s-3/2}{\left(s-3/2\right)^2-1/4}\right)+4L^{-1}\left(\frac{1}{\left(s-3/2\right)^2-1/4}\right)$$

$$=2e^{-t}+2e^{\left(\frac{3}{2}\right)t}L^{-1}\left(\frac{s}{s^2-\left(1/2\right)^2}\right)+4e^{\left(\frac{3}{2}\right)t}\sin h\left(\frac{t}{2}\right)\cdot 2$$

$$=2e^{-t}+2e^{\left(\frac{3}{2}\right)t}\cos h\left(\frac{t}{2}\right)+8e^{3/2t}\sin h\left(\frac{t}{2}\right)$$

25. Convolution of two functions

If $f(t)$ and $g(t)$ are given functions, then the convolution of $f(t)$ and $g(t)$ is defined as $\int_0^t f(u)g(t-u)du$.

It is denoted by $f(t)*g(t)$.

25.1. Convolution Theorem

If $f(t)$ and $g(t)$ are functions defined for $t \geq 0$, then $L(f(t)*g(t)) = L(f(t))L(g(t))$

(ie) $L(f(t)*g(t)) = F(s)\cdot G(s)$

where $F(s) = L(f(t)), G(s) = L(g(t))$

Proof :

By definition of Laplace Transform,

We have $$L(f(t))*g(t) = \int_0^\infty e^{-st}\left\{f(t)*g(t)\right\}dt$$

$$= \int_0^\infty e^{-st}\left\{\int_0^t f(u)g(t-u)du\right\}dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(u)g(t-u)dudt$$

on changing the order of integration,

$$= \int_0^\infty f(u)\left\{\int_u^\infty e^{-st} g(t-u)du\right\}dt$$

Put $t-u=v$ when $t=u, \ v=0$

$dt=dv$ when $t=\infty, \ v=\infty$

$$L(f(t))*g(t) = \int_0^\infty f(u)\left\{\int_0^\infty e^{-s(u+v)} g(v)dv\right\}du$$

$$= \int_0^\infty f(u)e^{-su}\left\{\int_0^\infty e^{sV} g(v)dv\right\}du$$

$$= \int_0^\infty e^{su} f(u)du \int_0^\infty e^{-sv} g(v)dv$$

$$= \int_0^\infty e^{-st} f(t)dt \int_0^\infty e^{-st} g(t)dt$$

$$= L(f(t))L(g(t))$$

$$\therefore L(f(t))*g(t) = F(s)\cdot G(s)$$

Corollary :

Using the above theorem

We get,

$$L^{-1}(F(s)\cdot G(s)) = f(t)*g(t)$$
$$= L^{-1}(F(s)*L^{-1}(G(s))$$

Note :

$$f(t)*g(t) = g(t)*f(t)$$

1. Find the value of $1*e^{-t}$

Solution :

Let $f(t)=1,\ \ g(t)=e^{-t}$

$f(u)=1,\ \ g(t-u)=e^{-(t-u)}$

$= e^{-t}e^{u}$

By definition, $f(t)*g(t)=\int_0^t f(u)g(t-u)du$

$$1*e^{t} = \int_0^t 1e^{-t}e^{u}du$$
$$= e^{-t}(e^{u})_0^t$$
$$= e^{-t}(e^{t}-1)$$
$$= 1-e^{-t}$$

2. Evaluate $1*\sin t$

Solution :

Let $f(t)=\sin t \quad g(t)=1$

$f(t)=\sin u \quad g(t-u)=1$

By definition,

$$f(t)*g(t)=\int_0^t f(u)g(t-u)du$$
$$t*e^{t} = \int_0^t \sin u 1\cdot du$$
$$= -(\cos u)_0^t$$
$$= -(\cos t-1)$$
$$= 1-\cos t$$

3. Evaluate $e^t * \cos t$

Solution :

Let $f(t) = \cos t \quad g(t) = e^t$

$f(t) = \cos u \quad g(t-u) = e^{t-u}$

$= e^t \cdot e^{-u}$

$$f(t) * g(t) = \int_0^t f(u) g(t-u) du$$

$$e^t * \cos t = \int_0^t \cos u e^t e^{-u} du$$

$$e^t * \cos t = e^t \int_0^t e^{-u} \cos u du$$

$$= e^t \left(\frac{e^{-u}}{(-1)^2 + 1^2} (-\cos u + \sin u) \right)_0^t$$

$$\left[\because \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) \right]$$

$$= e^t \left[\frac{e^{-t}}{2} (-\cos t + \sin t) - \frac{1}{2}(-1) \right]$$

$$= \frac{1}{2}(\sin t - \cos t) + \frac{1}{2} e^t$$

$$= \frac{1}{2}(\sin t - \cos t + e^t)$$

4. Use convolution theorem to find $L^{-1}\left(\frac{1}{(s+a)(s+b)} \right)$

Solution :

$$L^{-1}\left(\frac{1}{(s+a)(s+b)} \right) = L^{-1}\left(\frac{1}{(s+a)} \right) * L^{-1}\left(\frac{1}{(s+b)} \right)$$

$$= e^{-at} * e^{-bt}$$

$$= \int_0^t e^{-au} e^{-b(t-u)} du$$

$$= \int_0^t e^{-au} e^{-bt+bu} du$$

$$= e^{-bt} \left[\frac{e^{-(a-b)u}}{-(a-b)} \right]_0^t$$

$$= \frac{e^{-bt}}{-(a-b)} (e^{-(a-b)t} - 1)$$

$$= \frac{e^{-at}}{-(a-b)} + \frac{e^{-bt}}{(a-b)}$$

$$= \frac{1}{a-b} (e^{-bt} - e^{-at})$$

UNIT III

APPLICATIONS OF LAPLACE TRANSFORM

1.1 INTRODUCTION

Laplace transform is a powerful integral transform used to switch a function from the time domain to the s - domain. It can greatly simplify the solution of problems involving differential equations. It is very useful in obtaining solution of linear differential equations both ordinary and partial, solution of system of simultaneous differential equations, solution of integral equations and in the evaluation of definite integral.

Ordinary and partial differential equations describe the way certain quantities vary with time such as the current in an electrical circuit, the oscillations of a vibrating membrane, or the flow of heat through an insulated conductor these equations are generally coupled with initial conditions that describe the state of the system at time t = 0. A very powerful technique for solving these problems is that of Laplace transform which transform the differential equation into an algebraic equation from which we get the solution.

Solutions of Differential Equations using Laplace Transform

The following results will be used in solving differential and integral equations using Laplace transforms.

Theorem :

If $f(t)$ is continuous in $t \geq o, f'(t)$ is piecewise continuous in every finite interval in the range $t \geq o$ and $f(t)$ and $f'(t)$ are of exponential order, then

$$L(f'(t)) = sL(f(t)) - f(0)$$

Proof :

The given conditions ensure the existence of the Laplace transforms of $f(t)$ and $f'(t)$.

By definition, $L(f'(t)) = \int_0^\infty e^{-st} f'(t)dt$

$$= \int_0^\infty e^{-st} d(f(t))$$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty (-s)e^{-st} f(t)dt \text{, on integration by parts}$$

$$= \underset{t\to\infty}{Lim}\left[e^{-st} f(t)\right] - f(0) + s.L(f(t))$$

$$= 0 - f(0) + sL(f(t)) \qquad [\because f(t) \text{ is of exponential order}]$$

$$= sL(f(t)) - f(0)$$

Corollary 1

In the above theorem if we replace $f(t)$ by $f'(t)$ we get,

$$L(f''(t)) = sL(f'(t)) - f'(0)$$

$$= s\left[sL(f(t)) - f(0)\right] - f'(0)$$

$$= s^2 L(f(t)) - sf(0) - f'(0)$$

Repeated application of the above theorem gives the following result:

$$L(f^n(t)) = s^n L(f(t)) - s^{n-1} f(0) - s^{n-2} f^1(0) - ..... - f^{n-1}(0)$$

Solved Problems :

1. Using Laplace transform, solve $y' - y = t, y(0) = 0$.

Solution :

Given $y' - y = t, y(0) = 0$

Taking Laplace transfrom on both sides,

$$L(y') - L(y) = L(t)$$

$$sL(y) - y(0) - L(y) = \frac{1}{s^2}$$

$$L(y)\left[S - 1\right] = \frac{1}{s^2}$$

$$L(y) = \frac{1}{s^2(s-1)}$$

$$\therefore\ y = L^{-1}\left[\frac{1}{s^2(s-1)}\right]$$

$$y = \int_0^t\int_0^t L^{-1}\left(\frac{1}{s-1}\right) dt\ dt$$

$$y = \int_0^t\int_0^t e^t\ dt\ dt$$

$$= \int_0^t \left[e^t\right]_0^t dt$$

$$= \int_0^t \left[e^t - 1\right]_0^t dt$$

$$= \left(e^t - t\right)_0^t$$

$$= e^t - t - 1$$

2. Solve $y'' - 4y' + 8y = e^{2t}, y(0) = 2$ and $y'(0) = -2$.

Solution :

Taking Laplace transforms on the sides of the equation, we get

$$L(y'') - 4L(y') + 8L(y) = L(e^{2t})$$

$$\left[s^2L(y) - sy(0) - y^1(0)\right] - 4\left[sL(y) - y(0)\right] + 8L(y) = \frac{1}{s-2}$$

$$i.e., \left[s^2 - 4s + 8\right]L(y) = \frac{1}{s-2} + 2s - 10$$

$$L(y) = \frac{1}{(s-2)(s^2-4s+8)} + \frac{2s-10}{s^2-4s+8}$$

$$= \frac{A}{s-2} + \frac{Bs+C}{s^2-4s+8} + \frac{2s-10}{s^2-4s+8}$$

Solving we get $A=\frac{1}{4}, \quad B=\frac{-1}{4}, \quad C=\frac{1}{2}$

$$=\frac{\frac{1}{4}}{s-2}+\frac{\frac{-1}{4}s+\frac{1}{2}}{s^2-4s+8}+\frac{2s-10}{s^2-4s+8}$$
$$=\frac{\frac{1}{4}}{s-2}+\frac{\frac{7}{4}s-\frac{19}{2}}{s^2-4s+8}$$
$$=\frac{\frac{1}{4}}{S-2}+\frac{\frac{7}{4}(S-2)-6}{(S-2)^2+4}$$
$$y=\frac{1}{4}L^{-1}\left(\frac{1}{s-2}\right)+e^{2t}\left(\frac{\frac{7}{4}s-6}{s^2+4}\right)$$
$$=\frac{1}{4}e^{2t}+e^{2t}\left(\frac{7}{4}\cos 2t-3\sin 2t\right)$$
$$=\frac{1}{4}e^{2t}\left(1+7\cos 2t-12\sin 2t\right)$$

3. Use Laplace transform to solve $y'-y=e^t$ given that $y(0)=1$

Solution:

$$y'-y=e^t$$

Taking Laplace transform on both sides of the equation, we get $y'-y=t, y(0)=0$

$$\left[sL(y)-y(0)\right]-L(y)=\frac{1}{s-1}$$
$$L(y)[s-1]=\frac{1}{s-1}+1$$
$$L(y)=\frac{s}{(s-1)^2}$$
$$y=L^{-1}\left[\frac{s}{(s-1)^2}\right]$$
$$=L^{-1}\left[\frac{(s-1)+1}{(s-1)^2}\right]$$
$$=L^{-1}\left[\frac{1}{s-1}\right]+L^{-1}\left[\frac{1}{(s-1)^2}\right]$$
$$=e^t+te^t$$
$$=e^t(1+t)$$

4. Solve $\frac{d^2y}{dt^2}+9y=18t$ given that $y(0)=0=y\left(\frac{\pi}{2}\right)$

Solution :

$$y''+9y=18t \qquad \text{where} \quad y''=\frac{d^2y}{dt^2}$$

Taking Laplace transform on both sides of the equation, we get

$$L(y'')+9L(y)=18L(t)$$

$$\left[s^2L(y)-sy(0)-y'(0)\right]+9L(y)=\frac{18}{s^2}$$

$$L(y)\left[s^2+9\right]=\frac{18}{s^2}+y'(0)$$ [$\because y'(0)$ is not given we can take it to be a constant a]

$$=\frac{18}{s^2}+a$$

$$=\frac{as^2+18}{s^2}$$

$$L(y)=\frac{as^2+18}{s^2(s^2+9)}$$

$$=\frac{a}{s^2+9}+\frac{18}{s^2(s^2+9)}$$

$$y=L^{-1}\left(\frac{a}{s^2+9}\right)+L^{-1}\left(\frac{18}{s^2(s^2+9)}\right)$$

$$=L^{-1}\left(\frac{a}{s^2+9}\right)+L^{-1}\left(\frac{2}{s^2}-\frac{2}{(s^2+9)}\right)$$ (using partial fractions)

$$=\frac{a\sin 3t}{a}+2t-\frac{2\sin 3t}{3}$$

Now, using the conditions $y=0$ and $t=\frac{\pi}{2}$ we have

$$0 =\frac{a}{3}\sin\left(\frac{3\pi}{2}\right)+\pi-\frac{2}{3}\sin\left(\frac{3\pi}{2}\right)$$

$$=-\frac{a}{3}+\pi+\frac{2}{3}$$

$$\frac{a}{3}=\frac{3\pi+2}{3}$$

Hence $a=3\pi+2$

$$\therefore y=\frac{(3\pi+2)\sin 3t}{3}+2t-\frac{2\sin 3t}{3}$$

$$=\pi\sin 3t+2t$$

Solution of Integral equations using Laplace transform

Theorem :

If $f(t)$ is piecewise continuous in every finite interval in the range $t \geq 0$ and is of the exponential order, then

$$L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L(f(t))$$

Proof :

Let $g(t) = \int_0^t f(t)dt$

$\therefore \quad g^1(t) = f(t)$

$\therefore \quad L(g^1(t)) = sL(g(t)) - g(0)$

$i.e \quad L(f(t)) = sL\left(\int_0^t f(t)dt\right) - \int_0^0 f(t)dt$

$\therefore \quad L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L(f(t))$

Corollary :

$$L\left[\int_0^t \int_0^t f(t)dtdt\right] = \frac{1}{s^2}L(f(t))$$

In generl

$$L\left[\int_0^t \int_0^t \ldots\ldots \int_0^t f(t)(dt)^n\right] = \frac{1}{s^n}L(f(t))$$

Problems :

1. Solve $y + \int_0^t ydt = t^2 + 2t$

Solution :

Given $y + \int_0^t ydt = t^2 + 2t$

Taking Laplace Transform on both sides

$$L(y) + L\left(\int_0^t ydt\right) = L(t^2) + L(2t)$$

$$L(y) + \frac{1}{s}L(y) = \frac{2}{s^3} + \frac{2}{s^2}$$

$$L(y)\left[1 + \frac{1}{s}\right] = 2\left[\frac{1+s}{s^3}\right]$$

$$L(y)\left[\frac{s+1}{s}\right]=2\left[\frac{s+1}{s^3}\right]$$

$$L(y)=2\left[\frac{s+1}{s^3}\right]\left[\frac{s}{s+1}\right]$$

$$=\frac{2}{s^2}$$

$$y=L^{-1}\left(\frac{2}{s^2}\right)=2t$$

2. Solve $\frac{dy}{dt}+2y+\int_0^t ydt=2\cos t, \quad y(0)=1$

Solution :

Given $\quad y'+2y+\int_0^t ydt=2\cos t$

Taking Laplace Transform on both sides

$$L(y')+2L(y)+L\left(\int_0^t ydt\right)=2L(\cos t)$$

$$sL(y)-y(0)+2L(y)+\frac{1}{s}L(y)=\frac{2s}{s^2+1}$$

$$L(y)\left[s+2+\frac{1}{s}\right]-1=\frac{2s}{s^2+1}$$

$$L(y)\left[\frac{s^2+2s+1}{s}\right]=\frac{2s}{s^2+1}+1$$

$$L(y)=\left[\frac{s^2+2s+1}{s^2+1}\right]\left[\frac{s}{s^2+2s+1}\right]$$

$$=\frac{s}{s^2+1}$$

$$y=L^{-1}\left[\frac{s}{s^2+1}\right]=\cos t$$

3. Using Laplace Transform solve $y+\int_0^t y(t)dt=e^{-t}$

Solution :

Given $\quad y+\int_0^t y(t)dt=e^{-t}$

Taking Laplace transform on both sides,

$$L(y)+L\left(\int_0^t y(t)dt\right)=L(e^{-t})$$

$$L(y)+\frac{1}{s}L(y)=\frac{1}{s+1}$$

$$L(y)\left[1+\frac{1}{s}\right]=\frac{1}{s+1}$$

$$L(y)\left[\frac{s+1}{s}\right]=\frac{1}{s+1}$$

$$L(y)=\frac{s}{(s+1)^2}$$

$$y=L^{-1}\left(\frac{s}{(s+1)^2}\right)=L^{-1}\left(\frac{s+1-1}{(s+1)^2}\right)$$

$$=L^{-1}\left(\frac{1}{s+1}\right)-e^{-t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$y=e^{-t}-e^{-t}t$$

$$y=e^{-t}(1-t)$$

4. Using Laplace transform, solve $x+\int_0^t x(t)dt=\cos t+\sin t$

Solution :

$$x+\int_0^t x(t)dt=\cos t+\sin t$$

Taking Laplace transform on both sides,

$$L(x)+L\left(\int_0^t x(t)dt\right)=L(\cos t+\sin t)$$

$$L(x)\left[1+\frac{1}{s}\right]=\frac{s+1}{s^2+1}$$

$$L(x)\left[\frac{s+1}{s}\right]=\frac{s+1}{s^2+1}$$

$$L(x)=\left(\frac{s+1}{s^2+1}\right)\left(\frac{s}{s+1}\right)$$

$$L(x)=\frac{s}{s^2+1}$$

$$\therefore x=L^{-1}\left(\frac{s}{s^2+1}\right)=\cos t$$

Solving Integral Equations using convolution

Theorem :

By the definition of convolution, we have $f(t)*g(t)=\int_0^t f(u)g(t-u)du$

and by convolution theorem, $L(f(t)*g(t))=L(f(t))L(g(t))$

Problems :

1. Solve $y=1+2\int_0^t e^{-2u}y(t-u)\,du$ ______(1)

Solution :

$\int_0^t e^{-2u}y(t-u)\,du$ is of the form $\int_0^t f(u)g(t-u)du$ where $f(t)=e^{-2t},\ \ g(t)=y(t)$

Taking Laplace Transform on both sides of (1),

$$L(y)=L(1)+2L\left[\int_0^t e^{-2u}y(t-u)\,du\right]$$

$$=\frac{1}{s}+2L\left[e^{-2t}*y(t)\right]$$ (Definition of convolution)

$$=\frac{1}{s}+2L(e^{-2t})L(y)$$ (Convolution theorem)

$$=\frac{1}{s}+2\left(\frac{1}{s+2}\right)L(y)$$

$$L(y)=\frac{1}{s}+\frac{2}{s+2}L(y)$$

$$L(y)\left[1-\frac{2}{s+2}\right]=\frac{1}{s}$$

$$L(y)\left[\frac{s}{s+2}\right]=\frac{1}{s}$$

$$L(y)=\frac{s+2}{s^2}=\frac{1}{s}+\frac{2}{s^2}$$

$$y=L^{-1}\left(\frac{1}{s}+\frac{2}{s^2}\right)$$

$$y=1+2t$$

2. Using Laplace transform solve $y=1+\int_0^t y(u)\sin(t-u)du$

Solution :

Given $y=1+\int_0^t y(u)\sin(t-u)du$

Taking Laplace transform on both sides,

$$L(y) = L(1) + L\left[\int_0^t y(u)\sin(t-u)du\right] \qquad \_\_\_\_(1)$$

Now the integral $\int_0^t y(u)\sin(t-u)du$ is of the form $\int_0^t f(u)\,g(t-u)du$ where $f(t) = y(t), \quad g(t) = \sin t$

$\therefore$ (1) becomes

$$L(y) = \frac{1}{s} + L(y(t) * \sin t)$$

$$L(y) = \frac{1}{s} + L(y) \cdot \frac{1}{s^2+1}$$

$$L(y)\left[1 - \frac{1}{s^2+1}\right] = \frac{1}{s}$$

$$L(y)\left[\frac{s^2}{s^2+1}\right] = \frac{1}{s}$$

$$L(y) = \frac{s^2+1}{s^3}$$

$$= \frac{1}{s} + \frac{1}{s^3}$$

$$y = L^{-1}\left(\frac{1}{s}\right) + \frac{1}{2}L^{-1}\left(\frac{2}{s^3}\right)$$

$$y = 1 + \frac{1}{2}t^2$$

3. Using Laplace transform, solve $f(t) = \cos t + \int_0^t e^{-u} f(t-u)du$

Solution :

Given that $f(t) = \cos t + \int_0^t e^{-u} f(t-u)du \qquad \_\_\_\_\_(1)$

Taking Laplace transform on both sides of (1),

$$L(f(t)) = L(\cos t) + L\left[\int_0^t e^{-u} f(t-u)du\right]$$

$$= \frac{s}{s^2+1} + L(e^{-t} * f(t))$$

$$= \frac{s}{s^2+1} + L(e^{-t})L(f(t))$$

$$= \frac{s}{s^2+1} + \frac{1}{s+1}L(f(t))$$

$$L(f(t))\left[1-\frac{1}{s+1}\right]=\frac{s}{s^2+1}$$

$$L(f(t))\left[\frac{s}{s+1}\right]=\frac{s}{s^2+1}$$

$$L(f(t))=\frac{s+1}{s^2+1}$$

$$f(t)=L^{-1}\left(\frac{s}{s^2+1}\right)+L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$f(t)=\cos t+\sin t$$

4. Solve the integral equation $y(t)=t^2+\int_0^t y(t)\sin(t-u)du$

Solution :

$$y(t)=t^2+\int_0^t y(t)\sin(t-u)du$$

Taking Laplace transform obn both sides,

$$L(y(t))=L(t^2)+L\left[\int_0^t y(t)\sin(t-u)du\right]$$

$$L(y)=\frac{2}{s^3}+L(y(t)*\sin t)$$

$$=\frac{2}{s^3}+L(y)L(\sin t)$$

$$=\frac{2}{s^3}+L(y)\left(\frac{1}{s^2+1}\right)$$

$$L(y)\left(1-\frac{1}{s^2+1}\right)=\frac{2}{s^3}$$

$$L(y)\left(\frac{s^2}{s^2+1}\right)=\frac{2}{s^3}$$

$$L(y)=\frac{2(s^2+1)}{s^5}=\frac{2}{s^3}+\frac{2}{s^5}$$

$$y=L^{-1}\left(\frac{2}{s^3}\right)+\frac{2}{4!}L^{-1}\left(\frac{4!}{s^5}\right)$$

$$y=t^2+\frac{1}{12}t^4$$

Simulaneous differential equations

1. Using Laplace transform solve

$$\frac{dx}{dt}+y=\sin t$$
$$\frac{dy}{dt}+x=\cos t$$

given $x(0)=2$ and $y(0)=0$

Solution :

Applying Laplace transform to the given equations

We get, $L(x')+L(y)=L(\sin t)$

$L(y')+L(x)=L(\cos t)$

$$\therefore sL(x)-x(0)+L(y)=\frac{1}{s^2+1}$$
$$sL(y)-y(0)+L(x)=\frac{s}{s^2+1}$$
$$\therefore sL(x)+L(\text{y})=\frac{1}{s^2+1}+2$$
$$=\frac{2s^2+3}{s^2+1} \qquad \text{\_\_\_\_(1)}$$

Also $sL(y)+L(x)=\frac{s}{s^2+1}$ _____(2)

$$(1)\times s \Rightarrow s^2l(x)+sL(y)=\frac{(2s^2+3)s}{s^2+1} \qquad \text{\_\_\_\_\_(3)}$$
$$(2)\Rightarrow \quad L(x)+sL(y)=\frac{s}{s^2+1} \qquad \text{\_\_\_\_\_(4)}$$
$$(3)\text{ - }(4)\ \ (s^2-1)L(x)=\frac{(2s^2+3)}{s^2+1}-\frac{s}{s^2+1}$$
$$=\frac{2s^3+2s}{s^2+1}$$
$$L(x)=\frac{2s}{s^2-1} \qquad \text{\_\_\_\_\_(5)}$$

Substituting (5) in (2), we get

$$sL(y)=\frac{s}{s^2+1}-\frac{2s}{s^2-1}=\frac{s(s^2-1)-2s(s^2+1)}{(s^2+1)(s^2-1)}$$
$$=\frac{-s^3-3s}{(s^2+1)(s^2-1)}$$
$$=\frac{-s(s^2+3)}{-(s^2+1)(1-s^2)}$$
$$L(y)=\frac{(s^2+3)}{(s^2+1)(1-s^2)} \qquad \text{\_\_\_\_\_(6)}$$

From (5), $x = L^{-1}\left(\frac{2s}{s^2-1}\right)$

$= 2\cosh t$

$$y = L^{-1}\left(\frac{(s^2+3)}{(1-s^2)(s^2+1)}\right)$$

Consider $\frac{(s^2+3)}{(1-s^2)(s^2+1)} = \frac{A}{1-s} + \frac{B}{1+s} + \frac{Cs+D}{s^2+1}$ ____(7)

$$s^2+3 = A(1+s)(s^2+1) + B(1-s)(s^2+1) + (Cs+D)(1-s)(1+s)$$

Put $s=1,\quad 4 = A(2)(2)$

$\Rightarrow 4 = 4A \Rightarrow A = 1$

Put $s=-1,\quad 4 = B(2)(2)$

$\Rightarrow B = 1$

Put $s=0,\quad 3 = A+B+D$

$3 = 1+1+D$

$\Rightarrow D = 1$

Comparing the coefficient of S,

$0 = A - B + C$

$\Rightarrow C = 0$

Substituting the values of A, B, C, D in (7) we get

$$\frac{(s^2+3)}{(1-s^2)(s^2+1)} = \frac{1}{1-s} + \frac{1}{1+s} + \frac{1}{s^2+1}$$

$$\therefore y = L^{-1}\left(\frac{1}{1-s}\right) + L^{-1}\left(\frac{1}{1+s}\right) + L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$y = -e^{-t} + e^{-t} + \sin t$$

Hence the solution is $x = 2\cos ht$ and $y = -e^{-t} + e^{-t} + \sin t$

2. Solve $\frac{dx}{dt} + ax = y$

$\frac{dy}{dt} + ay = x$

given that $x = 0$ and $y = 1$ when $t = 0$

Solution :

Applying Laplace transform we get

$$L(x') + aL(x) = L(y)$$
$$L(y') + aL(y) = L(x)$$
$$\therefore sL(x) - x(0) + aL(x) = L(y)$$
$$sL(y) - y(0) + aL(y) = L(x)$$

Given that $x(0) = 0, \quad y(0) = 1$

$$\therefore sL(x) - x(0) + aL(x) = L(y)$$
$$sL(y) - y(0) + aL(y) = L(x)$$
$$\therefore sL(x) + aL(x) = L(y)$$
$$sL(y) - 1 + aL(y) = L(x)$$
$$\therefore (s+a)L(x) = L(y)$$
$$(s+a)L(x) - L(y) = 0 \qquad \_\_\_\_(1)$$
$$-L(x) + (s+a)L(y) = 1 \qquad \_\_\_\_(2)$$
$$(1) + (s+a) \times (2) \Rightarrow L(y)\left[(s+a)^2 - 1\right] = s + a$$
$$\therefore L(y) = \frac{s+a}{(s+a)^2 - 1}$$

Also by (1) $L(x) = \dfrac{1}{(s+a)^2 - 1}$

$$\therefore x = L^{-1}\left(\frac{1}{(s+a)^2 - 1}\right)$$
$$= e^{-at} L^{-1}\left(\frac{1}{s^2 - 1}\right)$$
$$= e^{-at} \sin ht$$
$$y = L^{-1}\left(\frac{s+a}{(s+a)^2 - 1}\right)$$
$$= e^{-at} L^{-1}\left(\frac{s}{s^2 - 1}\right)$$
$$= e^{-at} \cos ht$$

UNIT IV

FOURIER TRANSFORM

Fourier Transforms

Complex Fourier Transform (Infinite)

Let $f(x)$ be a function defined in $(-\infty,\infty)$ $f : R \to C$ and be piece-wise continuous in each finite partial interval then the complex Fourier transform of $f(x)$ is defined by

$$F(s) = F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx}dx$$

Inverse Fourier Transform

Inverse complex Fourier transform of F(s) is given by

$$f(x) = F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)e^{-isx}ds$$

Properties of Fourier Transforms

1. Linearity property

If $F(s)$ and $G(s)$ are the Fourier transforms of $f(x)$ and $g(x)$, then

$F[af(x) + bg(x)] \quad = aF[f(x)] + bF[g(x)]$

2. Shifting property

If $F[f(x)] = F(s)$ then $F(f(x - a) = e^{ias} F[f(x)] = e^{ias} F(s)$

3. Changeof scale property

If $F[f(x)] = F(s)$ then $F[f(ax)] = \frac{1}{|a|} F\left(\frac{s}{a}\right)$ where $a \neq 0$

4. Shifting in s

If $F[f(x)] = F(s)$ then $F(e^{iax} f(x)) = F(s+a)$

5. Modulation Property

If $F(f(x)) = F(s)$ then $F[\cos ax f(x)] = \frac{1}{2}[F(s+a)+F(s-a)]$

6. Fourier transform of Derivative

If $F[f(x)] = F(s)$ and derivative $f'(x)$ is continuous, absolutely integrable on $(-\infty, \infty)$, then $F[f'(x)] = -(is)\, F(s)$ if $f(x) \to 0$ as $x \to \pm\infty$

7. Derivative of transform

If $F[f(x)] = F(s)$, then $F(x^n f(x)) = (-i)^n \dfrac{d^n F(s)}{ds^n}$

Definition: Convolution of two functions.

The convolution of two functions $f(x)$ and $g(x)$ is defined as

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) g(x-t) dt$$

PROBLEMS

Problem 1. Find the Fourier transform of $f(x) = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}$. Hence evaluate $\int_0^{\infty} \frac{\sin s}{s} ds$.

Solution: Fourier transform of $f(x)$ is $F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \cos sx\, dx \qquad (\because \sin sx \text{ is an odd } fn.)$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{a} \cos sx\, dx$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \left[\frac{\sin sx}{s} \right]_0^a$$

$$F(s) = \frac{\sqrt{2}}{\sqrt{\pi}} \left[\frac{\sin as}{s} \right]$$

By inverse Fourier transforms,

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{\sin as}{s}(\cos sx-i\sin sx)\,ds$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin as}{s}\cos sx\,ds \quad \left[\because \frac{\sin as}{s}\sin\ sx\ is\ odd\right]$$

$$\text{f(x)} = \frac{2}{\pi}\int_{0}^{\infty}\left(\frac{\sin as}{s}\right)\cos sx\,ds$$

Put $a = 1$, $x = 0$

$$f(0)=\frac{2}{\pi}\int_{0}^{\infty}\frac{\sin s}{s}\,ds$$

$$\frac{\pi}{2}\times 1=\int_{0}^{\infty}\frac{\sin s}{s}ds\ (\therefore\ f(x)=1,-a\le x\le a)$$

$$\therefore\int_{0}^{\infty}\frac{\sin s}{s}ds=\frac{\pi}{2}$$

Definition: If the fourier transform of f(x) is equal to f(s) then the function f(x) is called **self-reciprocal**. i.e. F(f(x)) = f(s)

Problem 2: Find the Fourier transform of $e^{-a^2x^2}$. Hence prove that $e^{\frac{-x^2}{2}}$ is self-reciprocal with respect to Fourier Transforms.

Solution:

$$F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-a^2x^2}e^{isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-(a^2x^2)+isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-(a^2x^2-isx)}dx \qquad \dots (1)$$

Consider $a^2x^2 - isx = (ax)^2 - 2(ax)\frac{(is)}{2a} + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2$

$$= \left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2} \qquad \ldots (2)$$

Substitute (2) in (1), we get

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(ax - \frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-t^2} \frac{dt}{a} \qquad \text{Let } t = ax - \frac{is}{2a}, dt = adx$$

$$F[e^{-a^2x^2}] = \frac{1}{a\sqrt{2\pi}} e^{-\frac{s^2}{4a^2}} \sqrt{\pi} \qquad \left[\therefore \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}\right]$$

$$F[e^{-a^2x^2}] = \frac{1}{a\sqrt{2}} e^{\frac{s^2}{4a^2}} \qquad \ldots (3)$$

Put $a = \frac{1}{\sqrt{2}} in$ (3)

$$F[e^{-x^2/2}] = e^{-s^2/2}$$

$\therefore e^{-s^2/2}$ is self-reciprocal with respect to Fourier Transform.

Problem 3: State and Prove convolution theorem on Fourier transform.

Solution:

Statement: If $F(s)$ and $G(s)$ are Fourier transform of $f(x)$ and $g(x)$ respectively, Then the Fourier transform of the convolutions of $f(x)$ and $g(x)$ is the product of their Fourier transforms.

i.e. $F[f(x)*g(x)] = F[f(x)]F[g(x)] = F(s)G(s)$

Proof:

$$F(f*g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f*g)(x) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) g(x-t) dt\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-t)\, e^{isx} dx dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} g(x-t)\, e^{isx}\, dx \right) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) F(g(x-t) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, F(g(t)) dt \qquad [\therefore f(g(x-t) = e^{ist} F(g(t))]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt G(s) \qquad [\therefore F(g(t)) = G(s)]$$

$$F(f*g) = F(s).G(s). \qquad [\therefore F(f(t)) = F(s)].$$

Problem 4: Find the Fourier transform of $f(x) = \begin{cases} a^2 - x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$ and hence evaluate

(i) $\int_0^{\infty} \left(\frac{\sin t - t\cos t}{t^3} \right) dt$ **(ii)** $\int_0^{\infty} \left(\frac{\sin t - t\cos t}{t^3} \right)^2 dt$

Solutions:

Fourier transform of $f(x)$ is

$$F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x).e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 + \int_{-a}^{a} (a^2 - x^2) e^{isx} dx + 0 \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{a} (a^2 - x^2)(\cos sx + i \sin x)\, dx \right]$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{a} (a^2 - x^2) \cos sx\, dx \left[\therefore (a^2 - x^2) \sin sx \textit{ is an odd fn.} \right]$$

$$= \sqrt{\frac{2}{\pi}}\left[(a^2 - x^2)\left(\frac{\sin sx}{s}\right) - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (2)\left(\frac{\sin sx}{s^3}\right)\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[0 - \frac{2a\cos as}{s^2} + \frac{2\sin as}{s^3}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{-2as\cos as + 2\sin as}{s^3}\right]$$

$$F(s) \quad = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin as - as\cos as}{s^3}\right] \qquad \ldots (1)$$

By inverse Fourier transforms,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)e^{-isx}dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right)(\cos sx - i\sin sx)\, ds$$

$$f(x) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin as - as\cos as}{s^3}\cos sx\, dx$$ (the second term is on odd function)

$$f(x) = \frac{4}{\pi} \int_{0}^{\infty} \frac{\sin as - as\cos as}{s^3}\cos sx\, dx$$

Put $a = 1$

$$f(x) = \frac{4}{\pi} \int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3}\cos sx\, dx \qquad \left[f(x) = \begin{cases} 1 - x^2, & |x| \le 1 \\ 0, & |x| \ge 1 \end{cases}\right]$$

Put $x = 0$

$$f(0) = \frac{4}{\pi} \int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3}\, dx \qquad \left[\begin{aligned} f(0) &= 1 - 0 \\ &= 1 \end{aligned}\right]$$

$$1 = \frac{4}{\pi} \int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3}\, ds$$

$$\int_{0}^{\infty} \frac{\sin t - t\cos t}{t^3}\, dt = \frac{\pi}{4}$$ [by changing s →t]

Using Parseval's identify

$$\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

$$\int_{-\infty}^{\infty} \left[2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right)\right]^2 ds = \int_{-\infty}^{\infty} |a^2 - x^2|^2\, dx$$

$$\int_{-\infty}^{\infty} \frac{8}{\pi}\left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \int_{-1}^{1} (1-x^2)^2\, dx \text{ (put a = 1)}$$

$$2\times\frac{8}{\pi}\int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\int_{0}^{1} (1-x^2)^2\, dx$$

$$\frac{16}{\pi}\int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = 2\left[x + \frac{x^5}{5} - \frac{2x^3}{3}\right]_0^1$$

$$\int_{0}^{\infty} \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \frac{\pi}{16}\times 2\left(\frac{8}{15}\right) = \frac{\pi}{15}$$

Put s = t, $\int_{0}^{\infty} \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}.$

Problem 5: Find the Fourier transform of $f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & |x|\geq 1 \end{cases}$ and hence find the value of **(i)** $\int_{0}^{\infty} \frac{\sin^2 t}{t^2}\, dt.$ **(ii)** $\int_{0}^{\infty} \frac{\sin^4 t}{t^4}\, dt.$

Solution:

The Fourier transform of $F(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\, dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} (1-|x|)(\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} (1-|x|)\cos sx\, dx \;[\because (1-|x|)\sin sx \textit{ is an odd fn.}]$$

$$= \frac{2}{\sqrt{2\pi}}\left\{(1-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(\frac{\cos sx}{s^2}\right)\right\}_0^1$$

$$= \frac{2}{\sqrt{2\pi}}\left\{-\frac{\cos s}{s^2}+\frac{1}{s^2}\right\}$$

$$F(s) = \sqrt{\frac{2}{\pi}}\left[\frac{1-\cos s}{s^2}\right] \quad (1)$$

(i) By inverse Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}}\left[\frac{1-\cos s}{s^2}\right](\cos sx - i\sin sx)ds \text{ (by (1))}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1-\cos s}{s^2}\right)\cos sx\, ds \text{ (Second term is odd)}$$

$$f(x)= \frac{2}{\pi}\int_{0}^{\infty}\left(\frac{1-\cos s}{s^2}\right)\cos sx\, ds$$

Put $x = 0$

$$f(0)= 1-|0| = \frac{2}{\pi}\int_{0}^{\infty}\left(\frac{1-\cos s}{s^2}\right)ds$$

$$\int_{0}^{\infty}\left(\frac{1-\cos s}{s^2}\right)ds = \frac{\pi}{2}$$

$$\int_{0}^{\infty}\frac{2\sin^2(s/2)}{s^2}ds = \frac{\pi}{2}$$

Put $t = s/2$ $\quad ds = 2dt$

$$\int_{0}^{\infty}\frac{2\sin^2 t}{(2t)^2}2dt = \frac{\pi}{2}$$

$$\int_{0}^{\infty}\frac{\sin^2 t}{t^2}dt = \frac{\pi}{2}.$$

(ii) Using Parseval's identity.

$$\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

$$\int_{-\infty}^{\infty}\left[\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos s}{s^2}\right)\right]^2 ds = \int_{-1}^{1}(1-|x|)^2\,dx$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{1-\cos s}{s^2}\right)^2 ds = \int_{-1}^{1}(1-|x|)^2\,dx$$

$$\frac{4}{\pi}\int_{0}^{\infty}\left(\frac{1-\cos s}{s^2}\right)^2 ds = 2\int_{0}^{1}(1-x)^2\,dx$$

$$\frac{4}{\pi}\int_{0}^{\infty}\left(\frac{2\sin^2\left(\frac{s}{2}\right)}{s^2}\right)^2 ds = \left[2\left(\frac{1-x}{-3}\right)^3\right]_0^1$$

$$\frac{16}{\pi}\int_{0}^{\infty}\left(\frac{\sin^2\left(\frac{s}{2}\right)}{s^2}\right)^4 ds = \frac{2}{3}; Let\ t = s/2, dt = \frac{ds}{2}$$

$$\frac{16}{\pi}\int_{0}^{\infty}\left(\frac{\sin t}{2t}\right)^4 2dt = \frac{2}{3}$$

$$\frac{16}{16\pi}\int_{0}^{\infty}\left(\frac{\sin t}{2t}\right)^4 dt = \frac{1}{3}$$

$$\int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}.$$

i.e. $\int_{-\infty}^{\infty} |f(t)|^2\ dx = \int_{-\infty}^{\infty} |F(s)|^2\ ds$

Fourier Sine and Cosine Transform

Infinite Fourier Sine Transform of f(x) is denoted by F_s{f(x)] and is defined as

$$F_s[f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\sin sx\,dx$$

Inverse Fourier Sine Transform is

$$f(x) = F^{-1}[F_s(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$$

The Fourier cosine Integral of f(x) in (0,∞) is

$$F_C[f(x)] = F_C(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

This is known as **Infinite Fourier Cosine Transform** of f(x).

The **Inverse Fourier Cosine Transform** is

$$f(x) = F^{-1}[F_C(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty F_C(s) \cos sx \, ds$$

Properties of Fourier Sine and Cosine Transforms

1. Linearity Property

(i) $F_c[af(x) + bg(x)] = aF_c[f(x)] + bF_c[g(x)]$

(ii) $F_s[af(x) + bg(x)] = aF_s[f(x)] + bF_s[g(x)]$ where a and b are constants.

2. Modulation property

If $F_c[f(x)] = F_c[s]$ and $F_s[f(x)] = F_s[s]$, then

(i) $F_c[f(x) \cos ax] = \frac{1}{2}[F_c(s+a) + F_c(s-a)]$

(ii) $F_s[f(x) \cos ax] = \frac{1}{2}[F_s(s+a) + F_s(s-a)]$

(iii) $F_c[f(x) \sin ax] = \frac{1}{2}[F_s(s+a) - F_s(s-a)]$

(iv) $F_s[f(x) \sin ax] = \frac{1}{2}[F_c(s-a) - F_c(s+a)]$

3. Change of Scale Property

(i) $F_C[f(ax)] = \frac{1}{a} F_C\left(\frac{s}{a}\right) \quad if \ a > 0$ (ii) $F_S[f(ax)] = \frac{1}{a} F_S\left(\frac{s}{a}\right) \quad if \ a > 0$

4. Differentiation of sine and cosine transform

(i) $F_C[xf(x)] = \frac{d}{ds}[F_S(s)] = \frac{d}{ds}[F_S(f(x))]$

(ii) $F_S[xf(x)] = -\frac{d}{ds}[F_C(s)] = -\frac{d}{ds}[F_C(f(x))]$

5. Identities

If $F_c(s)$ and $G_c(s)$ are the Fourier cosine transforms and $F_s(s)$ and $G_s(s)$ are the Fourier sine transforms of $f(x)$ and $g(x)$ respectively then

i) $$\int_0^\infty f(x)g(x)dx = \int_0^\infty F_c(s)G_c(s)ds$$

ii) $$\int_0^\infty f(x)g(x)dx = \int_0^\infty F_s(s)G_s(s)ds$$

iii) $$\int_0^\infty |f(x)|^2\, dx = \int_0^\infty |F_c(s)|^2\, ds = \int_0^\infty |F_s(s)|^2\, ds$$

Problem 1: Find the Fourier cosine and sine transformation of $f(x) = e^{-ax}$, $a > 0$. Hence deduce that $\int_0^\infty \frac{x \sin \alpha\, x}{1+x^2} dx = \frac{\pi}{2} e^{-\alpha}$. and $\int_0^\infty \frac{\cos xt}{a^2+t^2} dt = \frac{\pi}{2a} e^{-a|x|}$

Solution:

The Fourier cosine transform is $F_C[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx dx$

$$\Rightarrow F_C[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2} (-a \cos sx + s \sin sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{1}{a^2+s^2} (-a+0) \right] = \sqrt{\frac{2}{\pi}} . \frac{a}{a^2+s^2}$$

The Fourier sine transform is $F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx\, dx$

$$F_s(e^{-ax}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}(-a\sin sx - s\cos sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[0 - \frac{1}{a^2+s^2}(0-s)\right] = \sqrt{\frac{2}{\pi}}.\frac{s}{a^2+s^2}$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)$$

By inverse Sine transform, we get

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right)\sin sx\,ds$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{s\sin sx}{s^2+a^2}ds$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{s\sin sx}{s^2+a^2}ds$$

$$\frac{\pi}{2}f(x) = \int_0^\infty \frac{s\sin sx}{s^2+a^2}ds$$

$$\frac{\pi}{2}e^{-ax} = \int_0^\infty \frac{s\sin sx}{s^2+a^2}ds$$

Put $a = 1$, $x = \alpha$

$$\frac{\pi}{2}e^{-a} = \int_0^\infty \frac{s\sin sx}{s^2+1}ds$$

Replace 's' by 'x' and 'x' by 's'

$$\int_0^\infty \frac{x\sin sx}{1+x^2}dx = \frac{\pi}{2}e^{-\alpha}.$$

Using Fourier inverse cosine transform,

$$\text{f(x)} = e^{-a|x|} = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c(s)\cos sx ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + s^2} \right] \cos sx \, ds$$

$$= \frac{2a}{\pi} \int_0^\infty \frac{\cos sx}{a^2 + s^2} ds$$

$$= \frac{2a}{\pi} \int_0^\infty \frac{\cos xt}{a^2 + t^2} dt \qquad \text{(Replace ‘}s\text{’ by ‘}t\text{’)}$$

$$\int_0^\infty \frac{\cos xt}{a^2 + t^2} dt = \frac{\pi}{2a} e^{-a|x|}$$

Problem 2: Find the Fourier cosine transform of $f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x \geq a \end{cases}$

Solution:

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx = \sqrt{\frac{2}{\pi}} \int_0^a \cos x \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a \left[\frac{\cos(s+1)x + \cos(s-1)x}{2} \right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)x}{s+1} + \frac{\sin(s-1)x}{s-1} \right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{\sin(s+1)a}{s+1} + \frac{\sin(s-1)a}{s-1} \right], \text{ provided } s \neq 1, s \neq -1.$$

Problem 3: Find the Fourier cosine transform of f(x) defined as

$$f(x) = \begin{cases} x & for \quad 0 < x < 1 \\ 2 - x & for \quad 1 < x < 2 \\ 0 & for \quad x > 2 \end{cases}$$

Solution: By definition of Fourier Cosine Transform

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^1 x\cos sx\,dx + \int_1^2 (2-x)\cos sx\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(x\frac{\sin sx}{s} - \frac{\cos sx}{s^2}\right)_0^1 + \left((2-x)\frac{\sin sx}{s} + \frac{\cos sx}{s^2}\right)_1^2\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(\frac{\sin s}{s} - \frac{\cos s - \cos 0}{s^2}\right) + \left(0-(1)\frac{\sin s}{s} + \frac{\cos 2s - \cos s}{s^2}\right)\right]$$

$$F_c(s) = \sqrt{\frac{2}{\pi}}\left[\frac{\cos 2s - 2\cos s + 1}{s^2}\right]$$

Problem 4: Find the Fourier sine transform of $\frac{1}{x}$.

Solution:

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$F_s\left(\frac{1}{x}\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}\sin sx\,dx$$

Let $sx = \theta$, $sdx = d\theta$;

X	0	∞
θ=sx	0	∞

$$F_s\left(\frac{1}{x}\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{s}{\theta}\sin\theta\frac{d\theta}{s}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin\theta}{\theta}d\theta\left[\therefore \int_0^\infty \frac{\sin\theta}{\theta}d\theta = \frac{\pi}{2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2}\right) = \sqrt{\frac{\pi}{2}}.$$

Problem6: Using Parseval's Identity calculate

(a) $\int_0^\infty \frac{1}{(a^2+x^2)^2}dx$ **(b)** $\int_0^\infty \frac{x^2}{(x^2+a^2)^2}dx$

Solution: (a) By Parseval's identity.

$$\int_0^\infty |f(x)|^2\,dx = \int_0^\infty |F_c(s)|^2\,ds$$

$$\int_0^\infty e^{-2ax}dx = \int_0^\infty \left[\sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2}\right]^2 ds$$

$$\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{2}{\pi}a^2\int_0^\infty \frac{ds}{(a^2+s^2)^2}$$

$$\frac{1}{2a} = \frac{2a^2}{\pi}\int_0^\infty \frac{ds}{a^2+s^2}$$

i.e. $\int_0^\infty \frac{dx}{(a^2+x^2)^2} = \frac{\pi}{4a^3}$ [Replace, s by x]

(b) By Parseval's identity.

$$\int_0^\infty |f(x)|^2\,dx = \int_0^\infty |F_s(f(x))|^2\,ds$$

$$\int_0^\infty (e^{-ax})^2 dx = \frac{2}{\pi}\int_0^\infty \left(\frac{s}{a^2+s^2}\right)^2 ds$$

i.e $\int_0^\infty \frac{s^2}{(a^2+s^2)^2}ds = \frac{\pi}{2}\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \frac{\pi}{2}\times\frac{1}{2a}$

$\int_0^\infty \frac{x^2}{(a^2+x^2)^2}dx = \frac{\pi}{4a}$ [Replace, s by x]

Problem 7. Evaluate **(a)** $\int_0^\infty \frac{1}{(x^2+1)(x^2+4)}dx$ **(b)** $\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx$, using Fourier cosine and sine transform.

Solution: **(a)** Let $f(x) = e^{-x}$ and $g(x) = e^{-2x}$

$$F_c(e^{-x}) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{s^2+1}(-\cos x + s\sin sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{1}{s^2+1}\right] \qquad \ldots (1)$$

$$F_c(e^{-2x}) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-2x}\cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{2}{s^2+4}\right) \qquad \dots (2)$$

$$\therefore \int_0^\infty f(x)g(x)dx = \int_0^\infty F_c(f(x))F_c(g(x))ds$$

$$\int_0^\infty e^{-x}e^{-2x}dx = \frac{2}{\pi}\int_0^\infty \left(\frac{1}{s^2+1}.\frac{2}{s^2+4}\right)ds \text{ (} from \text{ (1) \& (2))}$$

$$\int_0^\infty e^{-3x}dx = \frac{4}{\pi}\int_0^\infty \frac{ds}{(s^2+1)(s^2+4)}ds$$

$$\int_0^\infty \frac{ds}{(s^2+1)(s^2+4)} = \frac{\pi}{4}\left[\frac{e^{-3x}}{-3}\right]_0^\infty = \frac{\pi}{4}\left(\frac{1}{3}\right)$$

$$\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)} = \frac{\pi}{12} \text{ [Replace s to x]}$$

(b) To find $\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx.$

Let

$$f(x) = e^{-ax}, g(x) = e^{-bx}$$

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\, dx = \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+a^2}\right) \qquad \dots (1)$$

$$F_s(g(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-bx}\sin sx\, dx = \sqrt{\frac{2}{\pi}}\left(\frac{s}{s^2+b^2}\right) \qquad \dots (2)$$

$$\int_0^\infty f(x)g(x)dx = \int_0^\infty F_s[f(x)].F_s[g(x)]ds \text{ From (1) and (2)}$$

$$\int_0^\infty e^{-ax}e^{-bx}dx = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}ds$$

$$\int_0^\infty \frac{s^2}{(s^2+a^2)(s^2+b^2)}ds = \frac{\pi}{2}\int_0^\infty e^{-(a+b)x}dx$$

i.e $\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{2}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty = \frac{\pi}{2(a+b)}$ [Replace s to x]

Problem 7. Find the Fourier Cosine Transform of e^{-x^2} and hence Show that $xe^{\frac{-x^2}{2}}$ is self-reciprocal with respect to Fourier sine transform.

Solution

The Fourier Cosine Transform of f(x) is

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx dx$$

$$= \sqrt{\frac{2}{\pi}}.\frac{1}{2}.2\int_0^\infty e^{-x^2}\cos sx dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-x^2}e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{ R.P of } e^{-\frac{s^2}{4}}\int_{-\infty}^\infty \frac{e^{-x^2+isx}}{e^{\frac{-s^2}{4}}}dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{ R.P of } e^{-\frac{s^2}{4}}\int_{-\infty}^\infty e^{-x^2+isx+\frac{s^2}{4}}dx$$

$$= \frac{1}{\sqrt{2\pi}} \text{ R.P of } e^{-\frac{s^2}{4}}\int_{-\infty}^\infty e^{-\left(x-\frac{is}{2}\right)^2}dx$$

Put $x-\frac{is}{2} = y; \quad dx = dy$

When $x=-\infty, \quad y=-\infty$

$x=\infty, \quad y=\infty$

$$= \frac{1}{\sqrt{2\pi}} \text{ R.P of } e^{-\frac{s^2}{4}}\int_{-\infty}^\infty e^{-y^2}dy$$

$$= \frac{1}{\sqrt{2\pi}} \text{ R.P of } e^{-\frac{s^2}{4}}2\int_0^\infty e^{-y^2}dy$$

$$= \frac{1}{\sqrt{2\pi}} \text{ R.P of } e^{-\frac{s^2}{4}}2\frac{\sqrt{\pi}}{2} \qquad \left(\because \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}\right)$$

$$F_c\left(e^{-x^2}\right)=\frac{1}{\sqrt{2}}e^{\frac{-s^2}{4}}$$

Result : $F_s\left[xe^{\frac{-x^2}{2}}\right]=-\frac{d}{ds}F_c\left[e^{\frac{-x^2}{2}}\right]$

But $F_c\left[e^{\frac{-x^2}{2}}\right]=e^{\frac{-s^2}{2}}$

$$F_s\left[xe^{\frac{-x^2}{2}}\right]=-\frac{d}{ds}\left(e^{\frac{-s^2}{2}}\right)$$

$$=-e^{\frac{-s^2}{2}}.\left(\frac{-2s}{2}\right)$$

$$=se^{\frac{-s^2}{2}}$$

$\therefore\quad xe^{\frac{-s^2}{2}}$ is self reciprocal with respect to sine transform

Finite Fourier Transforms

If *f(x)* is a function defined in the interval (0,*l*)then **the finite Fourier sine transform** of *f(x)* in 0 <*x*<*l* is defined as

$F_S[f(x)]=\int_0^l f(x).\sin\frac{n\pi x}{l}dx$ where '*n*' is an integer.

The **inverse finite Fourier sine transform** of $F_s[f(x)]$ *f(x)* and given by

$$f(x)=\frac{2}{l}\sum_{n=1}^{\infty}F_s[f(x)]\sin\frac{n\pi x}{l}$$

The **finite Fourier cosine transform** of *f(x)* in 0 <*x* < *l* is defined as

$$F_C[f(x)]=\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

where *'n'* is an integer.

The **inverse finite Fourier cosine transform** of $F_c[f(x)]$ is *f(x)* and is given by

$$f(x)=\frac{1}{l}F_c(0)+\frac{2}{l}\sum_{n=1}^{\infty}F_c[f(x)]\cos\frac{n\pi x}{l}$$

Example 1. Find the finite Fourier sine and cosine transform of $f(x) = x^2$ in $0 < x < 1$.

Solution:

The finite Fourier sine transform is

$$F_S[f(x)] = \int_0^l f(x).\sin\frac{n\pi x}{l}\,dx$$

Here $f(x) = x^2$

$$\therefore \quad F_s[x^2] = \int_0^l x^2 \sin\frac{n\pi x}{l}\,dx$$

$$= \left[x^2\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 2x\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + \left(2\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{-l^3}{n\pi}\cos n\pi + \frac{2l^3}{n^3\pi^3}\cos n\pi - \frac{2l^3}{n^3\pi^3} \quad , \cos n\pi = (-1)^n, \ \sin n\pi = 0$$

$$= \frac{l^3}{n\pi}(-1)^{n+1} + \frac{2l^3}{n^3\pi^3}[(-1)^n - 1]$$

The finite Fourier cosine transform is

$$F_c[f(x)] = \int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

Here $f(x) = x^2$

$$\therefore \quad F_c[x^2] = \int_0^l x^2 \cos\frac{n\pi x}{l}\,dx$$

$$= \left[x^2\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 2x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2l^3}{n^2\pi^2}\cos n\pi \, , \cos n\pi = (-1)^n, \ \sin n\pi = 0$$

$$= \frac{2l^3}{n^2\pi^2}(-1)^n$$

Example 2: Find the finite Fourier sine and cosine transform of $f(x) = x$ in $(0, \pi)$

Solution:

The finite Fourier sine transform of $f(x)$ in $(0, \pi)$ is

$$F_S[f(x)] = \int_0^{\pi} f(x).\sin nx\, dx$$

Here $f(x) = x$ in $(0, \pi)$

$$\therefore \quad F_s[x] = \int_0^{\pi} x \sin nx\, dx$$

$$= \left[x\left(\frac{-\cos nx}{n}\right) - 1\frac{-\sin nx}{n^2} \right]_0^{\pi}$$

$$= -\frac{\pi}{n}\cos n\pi = (-)^{n+1}.\frac{\pi}{n}$$

The finite Fourier cosine transform of $f(x)$ in $(0, \pi)$ is

$$F_C[f(x)] = \int_0^{\pi} f(x).\cos nx\, dx$$

$$\therefore F_C[x] = \int_0^{\pi} (x)\cos nx\, dx$$

$$= \left[x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right) \right]_0^{\pi}$$

$$= \frac{1}{n^2}[\cos n\pi - 1]$$

$$= \frac{1}{n^2}[(-1)^n - 1]$$

Example 3: Find the finite Fourier sine and cosine transforms of $f(x) = e^{ax}$ in $(0,l)$

Solution:

We know that $F_S[f(x)] = \int_0^{l} f(x)\sin\frac{n\pi x}{l}\, dx$

Here $f(x) = e^{ax}$

$$\therefore F_s[e^{ax}] = \int_0^l e^{ax} \sin \frac{n\pi x}{l} dx$$

$$= \left\{ \frac{e^{ax}}{a^2 + \frac{n^2\pi^2}{l^2}} \left(a \sin \frac{n\pi x}{l} - \frac{n\pi}{l} . \cos \frac{n\pi x}{l} \right) \right\}_0^1$$

$$= \frac{e^{al}}{a^2 + \frac{n^2\pi^2}{l^2}} \left(\frac{-n\pi}{l} \cos n\pi \right) + \frac{\frac{n\pi}{l}}{a^2 + \frac{n^2\pi^2}{l^2}}$$

$$= \frac{n\pi l}{a^2 l^2 + n^2\pi^2} [(-1)^{n+1} e^{al} + 1]$$

$$F_C [e^{ax}] = \int_0^l f(x) \cos \frac{n\pi x}{l} dx$$

$$= \int_0^l e^{ax} \cos \frac{n\pi x}{l} dx$$

$$= \left\{ \frac{e^{ax}}{a^2 + \frac{n^2\pi^2}{l^2}} \left[a \cos \frac{n\pi x}{l} + \frac{n\pi}{l} \sin \frac{n\pi x}{l} \right] \right\}_0^l$$

$$= \frac{e^{al} . l^2}{a^2 l^2 + n^2\pi^2} (a \cos n\pi) - \frac{al^2}{a^2 l^2 + n^2\pi^2}$$

$$= \frac{al^2}{al^2 + n^2\pi^2} [e^{al} . (-1)^n - 1]$$

Example 4: Find the finite Fourier cosine transform of $f(x) = \sin ax$ in $(0, \pi)$.

Solution:

$$F_C[\sin ax] = \int_0^\pi \sin ax . \cos nx \, dx$$

$$= \frac{1}{2} \int_0^\pi [\sin(a+n)x + \sin(a-n)x] dx$$

$$= \frac{1}{2}\left[\frac{-\cos(a+n)x}{a+n} - \frac{\cos(a-n)x}{a-n}\right]_0^{\pi}$$

$$= \frac{-1}{2}\left[\frac{\cos(a+n)\pi}{a+n} + \frac{\cos(a-n)\pi}{a-n} - \frac{1}{a+n} - \frac{1}{a-n}\right]$$

$$= \frac{-1}{2}\left[\frac{(-1)^{a+n}}{a+n} + \frac{(-1)^{a-n}}{a-n} - \frac{1}{a+n} - \frac{1}{a-n}\right]$$

if both n and a are even

$$F_c(\sin ax) = \begin{cases} 0, & \textit{if both n and a are even} \\ \frac{1}{2}\left[\frac{2}{a+n} + \frac{2}{a-n\_}\right], & \textit{if n or a is odd} \end{cases}$$

$$F_c(\sin ax) = \begin{cases} 0, & \textit{if both n and a is odd} \\ \frac{2n}{a^2-n^2}, & \textit{if n or a is odd} \end{cases}$$