Unit-3:
Quantum Theory of Radiation

Spectral Distribution of Black Body Radiation:

All bodies emit heat radiations from their surfaces by virtue of their temperature. This radiation is called radiant energy. The nature of radiation depends upon the temperature. At a low temperature, a body emits radiations which are principally of longer wavelength. At a high temperature, the proportion of shorter wavelength radiation increases. It is of interest to see how the energy of total radiation from a hot body is distributed among different wavelengths at various temperatures. In other words, we would like to find out how radiant energy at wavelength λ depends upon frequency and temperature.

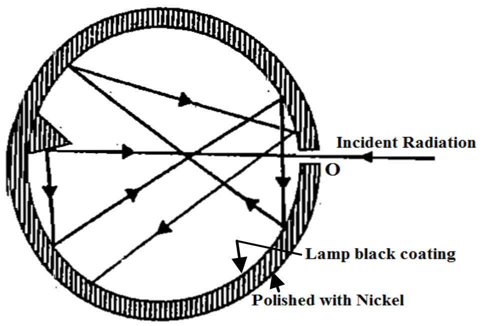

Fig.1: Fery Black Body

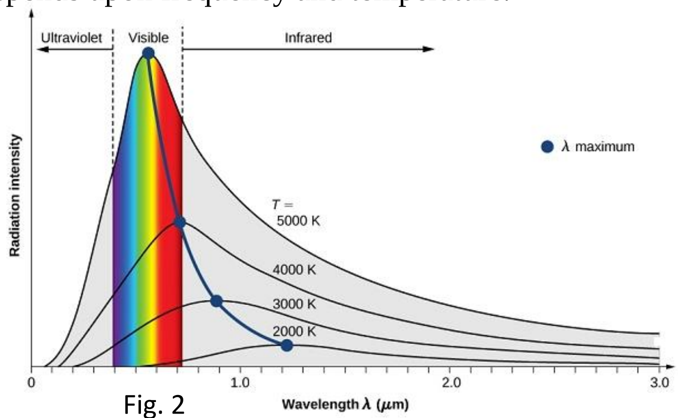

Fig. 2

Let us consider a hollow enclosure in the form of black body whose walls are maintained at a constant temperature T and a small hole is made in the enclosure as shown in fig.1. The radiations coming out are called **black body radiations**. These radiations are electromagnetic in nature and are the characteristics of that temperature. The radiations are supposed to consist of quanta or photons of energy $h\nu$, where h is the Planck constant and ν to frequency of radiation moving in all possible directions with the speed of light c. The momentum of the photon is equal to $\frac{h\nu}{c}$. The radiation inside the hollow enclosure consist of a very large number of photons of different energies as they have different frequencies or wavelengths and they can be supposed to form a **Photon gas**. These radiations had been experimentally studied for a long time but the graph depicting energy distribution between the energy and the mean wavelength of each range could not be explained theoretically on the basis of classical physics. Experimentally, the variation of $E(\lambda)$ with λ is obtained as shown in the fig.2.

In 1900, Max Planck gave the correct energy distribution on the basis of his quantum theory. Following are the results to explain the distribution of energy spectrum (spectral distribution of energy) of black body from the experimental curves of fig.2.

(i) Black body radiation may be regarded as a gas consisting of photons. These photons do not interact with one another so that the photon gas is an ideal gas.

(ii) Photons are particles of zero rest mass.

(iii) The number of photons inside the enclosure is very large and they have different energies.

(iv) The photons have integral angular momentum in units of $\frac{h}{2\pi}$.

(v) The distribution of energy of the spectrum of black body radiations in not uniform over a wide range of wavelengths i.e. energy is not uniformly distributed in the spectrum of a black body at a given temperature. In other words energy for different wavelengths is different.

(vi) For a given temperature, the energy of radiation emitted increases with the increase in wavelength and becomes maximum for a particular value of wavelength λ_m, and then starts decreasing with the further increase in wavelength.

(vii) An increase in temperature results in the increase in energy emission.

(viii) For very long wavelengths and for very short wavelengths, emission of energy is very small.

(ix) With increase in temperature, peak of the curve shifts towards shorter wavelength side.

(x) It has been observed that the product of maximum wavelength corresponding to maximum energy and absolute temperature for black body radiations is constant i.e. $\lambda_m T =$ constant, which is known as **Wein's displacement law**. Thus this confirms the **Wein's displacement law.**

(xi) Area under each curve represents the total energy emitted by the black body at a given temperature. This area increase with the increase of temperature and is found that the total energy emitted by the black body is directly proportional to the fourth power of the absolute temperature of the body i.e.

$$E \propto T^4 \Rightarrow E = \sigma T^4$$

This is known as **Stefan – Boltzmann's law** i.e. this verifies the **Stefan's law.**

Experimental arrangement:

Lummer and Pringsheim experimentally studied the distribution of energy among the radiation emitted by a black body at different temperatures. The experimental set up for the study of black body radiations is shown in fig.3. Their experimental black body was a small aperture of an electrical heated chamber whose temperature was measured by a thermocouple.

The radiations of an electrically heated black body (B) are collimated through a narrow slit and the collimated beam is incident on a stainless steel concave mirror M_1. The distance between the source and M_1 is kept equal to the focal length of M_1 to make the reflected beam parallel. After being reflected, the parallel beam of radiations is made to fall on one of the faces of a rock-salt or fluorspar prism ABC placed on the rotating (or turn) table of the spectrometer. The emergent beam consists of rays of different wavelengths and this emergent beam falls on another concave mirror M_2 and is focused on a line bolometer placed behind the slit S_2. The bolometer is connected to a sensitive galvanometer. The turn table is rotated slowly so that the different parts of the radiation spectrum successively fall on the bolometer and the corresponding deflection in galvanometer connected in the bolometer circuit are read (noted). The intensity of each line is proportional to the deflection in the galvanometer. A number of observations were taken by maintaining the black body at various temperatures ranging from 2000K to 6000K. A graph between the energy E_λ and the wavelength λ of the radiation is plotted at different temperatures at which the black body was maintained and the variation of $E(\lambda)$ with λ is obtained as shown in the spectrum fig.2.

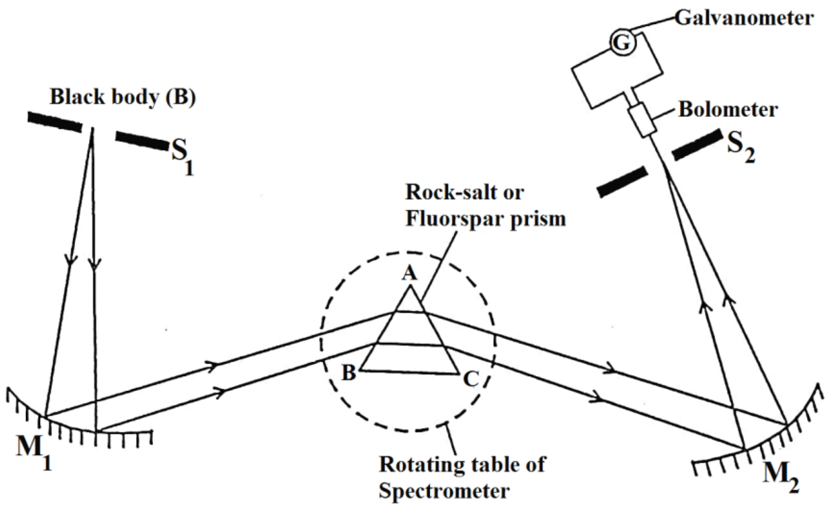

Fig. 3 **Lummer and Pringsheim Experiment.**

Planck's Quantum Postulates (Assumptions):

In Planck's quantum hypothesis the following assumptions are made:

1. A black body radiation chamber is filled up not only with radiation, but also with simple harmonic oscillators or resonators (energy emitters) of the molecular dimensions, known as Planck's oscillators or Planck's resonators, which can vibrate with all possible frequencies. The vibration of the resonator entails one degree of freedom only.
2. The oscillators or resonators cannot radiate or absorb energy continuously, but energy is emitted or absorbed in the form of packets or quanta called photons. Planck's assumed that each photon has an energy $h\nu$ where h is the Planck's constant, its value being equal to $6.625\times10^{-34}\,Joule-Sec$ and ν is the frequency of radiation. This assumption is the most revolutionary in character. In other word, the theory states that the exchange of energy between radiation and matter cannot take place continuously but only in certain multiples of the fundamental frequency of the **resonator** (energy emitter). As the energy of a photon is $h\nu$, the energy emitted or absorbed is equal to $0,\ h\nu,\ 2h\nu,\ 3h\nu\cdots\cdots\cdots nh\nu$, i.e., in multiples of some small unit, called as **quantum**.

Planck's Law of Blackbody Radiation:

The classical theory could not explain the distribution of energy spectrum of a black body. Max Planck a German physicist in 1900 made an important hypothesis known as Planck's hypothesis which states that

"A black body radiation cavity filled up not only with radiation but also filled with simple harmonic oscillator. Hence the exchange of radiation energy with matter does not take place continuously but discontinuously and discretely with an integral multiple of small unit of energy called the quantum or photon."

Max Planck suggested that the energy of a photon is directly proportional to the frequency ν of radiation. i.e.

$$E\propto\nu$$
$$E=h\nu$$

Where h is called Planck's constant and its value is $6.62\times10^{-34}\,JS$.

Since, an oscillator frequency ν can only emit or absorb the radiation in units of magnitude of $h\nu$ i.e.

$$E=nh\nu,\qquad n=1,2,3,4------$$

An oscillator can emit energy when it goes from one energy sate to another energy sate. The emission takes place only during the transition from a higher to lower energy sate.

Average energy of Planck oscillator:

The average energy per Planck oscillator is given by

$$\overline{E}=\frac{E}{N}\qquad\text{--- (1)}$$

Where, E is the total energy and N is the total number of Planck's oscillators.

Planck's oscillator is an oscillator which can absorb energy or emit energy only in amounts which are integral multiples of Planck's constant time the frequency ν of the oscillator.

Suppose, there are $N_0,\ N_1,\ N_2,\ N_3,\ -------\ N_n$ numbers of Planck's oscillators having energies $0,\ h\nu,\ 2h\nu,\ 3h\nu,--------nh\nu$ respectively then total numbers of oscillators are

$$N=N_0+N_1+N_2+N_3+--------+N_n$$

And total energy,

$$\therefore E=N_0\times0+N_1h\nu+N_2 2h\nu+N_3 3h\nu+--------+N_n nh\nu\qquad\text{--- (2)}$$

Since, accordance with Maxwell's distribution law, the distribution of N_n numbers of oscillators in nth state having energy $nh\nu$ is,

$$N_n = N_0 e^{-\frac{nh\nu}{k_B T}} \quad \text{--(3)} \qquad \left[\because \quad n_i = g_i e^{-\alpha} e^{-\beta E_i} = g_i e^{-\alpha} e^{-E_i/KT} = N_0 e^{-E_i/KT}\right]$$

Where, k_B is the Boltzmann constant.

Therefore,

The total number of Planck's oscillators in accordance with Maxwell's distribution law is given by

$$N = N_0 + N_0 e^{-\frac{h\nu}{k_B T}} + N_0 e^{-\frac{2h\nu}{k_B T}} + N_0 e^{-\frac{3h\nu}{k_B T}} + -------- + N_0 e^{-\frac{nh\nu}{k_B T}}, \qquad \text{Let, } e^{-\frac{h\nu}{k_B T}} = x$$

$$\Rightarrow N = N_0 \left(1 + x + x^2 + x^3 + ------- + x^n\right)$$

$$\Rightarrow N = N_0 \frac{1}{(1-x)} \qquad \because 1 + x + x^2 + x^3 + ------- + x^n = \frac{1}{(1-x)}$$

$$\Rightarrow N = N_0 \times \frac{1}{1 - e^{-\frac{h\nu}{k_B T}}} \quad \text{-- (4)}$$

The total energy E becomes

$$E = N_0 \times 0 + h\nu(N_0 e^{-\frac{h\nu}{k_B T}}) + 2h\nu(N_0 e^{-\frac{2h\nu}{k_B T}}) + 3h\nu(N_0 e^{-\frac{3h\nu}{k_B T}}) + -------- + nh\nu(N_0 e^{-\frac{nh\nu}{k_B T}})$$

$$\Rightarrow E = N_0 \left(h\nu\, e^{-\frac{h\nu}{k_B T}} + 2h\nu\, e^{-\frac{2h\nu}{k_B T}} + 3h\nu\, e^{-\frac{3h\nu}{k_B T}} + ------- + nh\nu\, e^{-\frac{nh\nu}{k_B T}} \right)$$

$$\Rightarrow E = N_0 \left(h\nu\, x + 2h\nu\, x^2 + 3h\nu\, x^2 + ------- + nh\nu\, x^n\right), \qquad \text{let } e^{-\frac{h\nu}{k_B T}} = x$$

$$\Rightarrow E = N_0 h\nu\, x \left(1 + 2x + 3x^2 + ------- + n\, x^{n-1}\right)$$

$$\Rightarrow E = N_0 h\nu\, x \frac{1}{(1-x)^2} \qquad \because 1 + 2x + 3x + ----- + nx^{n-1} = \frac{1}{(1-x)^2}$$

$$\Rightarrow E = N_0 h\nu\, e^{-\frac{h\nu}{k_B T}} \frac{1}{\left(1 - e^{-\frac{h\nu}{k_B T}}\right)^2} \quad \text{--- (5)}$$

Substituting the value of N and E from equation (4) and (5) in equation (1), we get

$$\bar{E} = \frac{E}{N} = \frac{N_0 h\nu\, e^{-\frac{h\nu}{k_B T}} \frac{1}{\left(1 - e^{-\frac{h\nu}{k_B T}}\right)^2}}{N_0 \times \frac{1}{1 - e^{-\frac{h\nu}{k_B T}}}}$$

$$\Rightarrow \bar{E} = \frac{h\nu\, e^{-\frac{h\nu}{k_B T}}}{1 - e^{-\frac{h\nu}{k_B T}}}$$

Now, dividing the numerator and denominator by $e^{-\frac{h\nu}{k_B T}}$, we get

$$\Rightarrow \bar{E} = \frac{h\nu}{\frac{1}{e^{-\frac{h\nu}{k_B T}}} - 1} \qquad \Rightarrow \bar{E} = \frac{h\nu}{e^{\frac{h\nu}{k_B T}} - 1} \quad \text{--- (6)}$$