Fluid Dynamics

A fluid is a substance which deforms continuously or flows when subjected to external shearing forces.

Fluid dynamics or Hydrodynamics is that branch of science which is concerned with the study of the motion of fluids or that of bodies in contact with fluids. Fluids are classified as liquids and gases. The former are not sensibly compressible except under the action of heavy forces whereas the latter are easily compressible and expand to fill any closed space.

Characteristics of Fluids (Liquid or Gas)

1. It has no definite shape of its own, but conforms to the shape of the containing vessel.
2. Even a small amount of shear force exerted on a fluid will cause it to undergo a deformation which continues as long as the force continues to be applied.
3. It is interesting to note that a solid suffers strain when subjected to shear forces whereas a fluid suffers rate of strain i.e. it flows under similar circumstances.

Some basic properties of fluid:

1. Density, Specific Weight and Specific Volume:

The density of a fluid is defined as the mass per unit volume. Mathematically, the density ρ at a point P may be defined as

$$\rho = \lim_{\delta v \to 0} \frac{\delta m}{\delta v}$$

Where, δv is the volume element around P and δm is the mass of the fluid within δv.

The specific weight γ of a fluid is defined as the weight per unit volume. Thus, $\gamma = \rho g$, where g is the acceleration due to gravity.

The specific volume of a fluid is defined as the volume per unit mass and is clearly the reciprocal of the density.

2. Pressure:

When a fluid is contained in a vessel, it exerts a force at each point of the inner side of the vessel. Such a force per unit area is known as pressure. Mathematically, the pressure p at a point P may be defined as

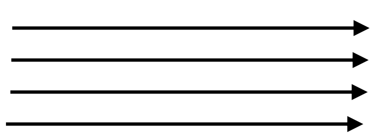
$$p = \lim_{\delta S \to 0} \frac{\delta F}{\delta S}$$

Where, δS is an elementary area around P and δF is the normal force due to fluid on δS.

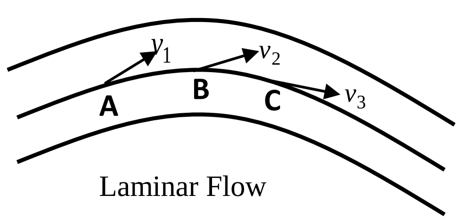
Types of fluid flows:

1. Laminar or Streamline flows:

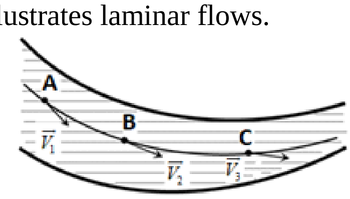
A flow, in which each fluid particle traces out a definite curve and the curves traced out by any two different fluid particles do not intersect, is said to be laminar. The following figure illustrates laminar flows.



Laminar Flow



Laminar Flow



Stream Line Flow

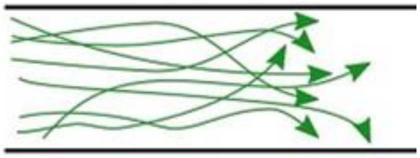
Let us consider a liquid flowing in a pipe. Let the velocity of flow be v_1 at A, v_2 at B and v_3 at C. If as time passes, the velocities at A, B and C are constant in magnitude and direction, then the flow is said to be steady. In a steady flow, each particle follows exactly the same path and has exactly the same velocity as its predecessor. In such a case, the liquid is said to have an orderly or streamline flow. **Thus, a liquid motion is called streamline motion when the velocity at every point in the liquid remains constant both in its magnitude and direction.**

The flow is steady or streamlined only as long as the velocity of the liquid does not exceed a limiting value, called the critical velocity. When the external pressure causing the flow of the liquid is excessive, the motion of the liquid takes place with a velocity greater than the critical velocity and the motion becomes unsteady or turbulent.

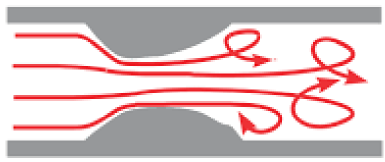
2. **Turbulent flow:**

A flow, in which each fluid particle does not trace out a definite curve and the curves traced out by fluid particles intersect, is said to be turbulent.

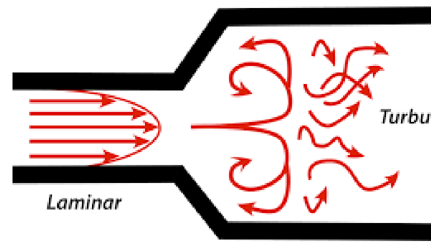
A liquid motion is called turbulent motion when the velocity at every point in the liquid is not constant and its magnitude is large. Further, the liquid moves in a zig-zag path. This disorderly motion takes place when the pressure difference between the ends to the tube is large. The following figure illustrates turbulent flows.



Turbulent Flow



Turbulent Flow



3. **Steady and unsteady flows:**

A flow, in which properties and conditions (Say P) associated with the motion of the fluid are independent of the time so that the flow pattern remains unchanged with time, is said to be steady. Mathematically, we may write $\partial P/\partial t = 0$.Here P may be velocity, density, pressure, temperature etc. On the other hand, a flow, in which properties and conditions associated with the motion of the fluid depend on the time so that the flow pattern varies with time, is said to be unsteady.

4. **Uniform and non-uniform flows:**

A flow, in which the fluid particles possess equal velocities at each section of the channel or pipe, is called uniform. On the other hand, a flow, in which the fluid particles possess different velocities at each section of the channel or pipe, is called non-uniform. These terms are usually used in connection with flow in channel.

5. **Rotational and Irrotational flows:**

A flow, in which the fluid particles go on rotating about their own axes, while flowing, is said to be rotational. On the other hand, a flow, in which the fluid particles do not rotate about their own axes, while flowing, is said to be irrotational.

6. **Barotropic flows:** The flow is said to be barotropic when the pressure is a function of the density.

Critical Velocity:

Critical velocity of a liquid is the velocity below which the motion of the liquid is orderly and above which the motion of the liquid becomes turbulent.

The expression for critical velocity is

$$v_c = \frac{K\eta}{\rho r}$$

Here, η is the coefficient of viscosity of the liquid, ρ is the density of liquid, r is the radius of the tube through which the liquid flows and K is constant, called Reynold's number. Its value is 1000 for narrow tubes. Reynold's number determines the nature of the liquid motion through a tube.

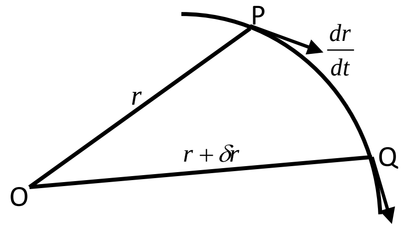
Velocity of fluid particle:

Let the fluid particle be at P at any time t and let it be at Q at time $t+\delta t$ such that $\overrightarrow{OP} = r$ and $\overrightarrow{OQ} = r + \delta r$. Then in the interval δt , the movement of the particle is $\overrightarrow{PQ} = \delta r$ and hence the velocity of the liquid particle q at P is given by

$$q = \lim_{\delta t \to 0} \frac{\delta r}{\delta t} = \frac{dr}{dt}$$

Clearly, q is a function of r & t .Hence it can be expressed as $q = f(r,t)$. If u, v, w are the components of q along the axes, we have

$q = u\hat{i} + v\hat{j} + w\hat{k}$



Acceleration of a fluid particle:

Let a fluid particle moves from $P(x, y, z)$ at time t to $Q(x+\delta x, y+\delta y, z+\delta z)$ at time $t+\delta t$. Let $q = u\hat{i} + v\hat{j} + w\hat{k}$ be the velocity of the fluid particle at P and $q+\delta q$ be the velocity of the same fluid particle at Q. Then, we have

$$\delta q = \frac{\partial q}{\partial x}\delta x + \frac{\partial q}{\partial y}\delta y + \frac{\partial q}{\partial z}\delta z + \frac{\partial q}{\partial t}\delta t$$

$$\Rightarrow \frac{\delta q}{\delta t} = \frac{\partial q}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial q}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial q}{\partial z}\frac{\delta z}{\delta t} + \frac{\partial q}{\partial t} \qquad \text{----- (1)}$$

Let, $\lim_{\delta t \to 0}\frac{\delta q}{\delta t} = \frac{Dq}{Dt}\, or\, \frac{dq}{dt}$, $\lim_{\delta t \to 0}\frac{\delta x}{\delta t} = \frac{dx}{dt} = u$, $\lim_{\delta t \to 0}\frac{\delta y}{\delta t} = \frac{dy}{dt} = v$ and $\lim_{\delta t \to 0}\frac{\delta z}{\delta t} = \frac{dz}{dt} = w$ ------ (2)

Now, making $\delta t \to 0$ and using equation (2), the equation (1) reduces to

$$a = \lim_{\delta t \to 0}\frac{\delta q}{\delta t} = \frac{\partial q}{\partial x}\lim_{\delta t \to 0}\frac{\delta x}{\delta t} + \frac{\partial q}{\partial y}\lim_{\delta t \to 0}\frac{\delta y}{\delta t} + \frac{\partial q}{\partial z}\lim_{\delta t \to 0}\frac{\delta z}{\delta t} + \frac{\partial q}{\partial t}$$

$$\Rightarrow a = \frac{dq}{dt} = \frac{Dq}{Dt} = u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} + w\frac{\partial q}{\partial z} + \frac{\partial q}{\partial t} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)q \qquad \text{---- (3)}$$

Let, $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

$$q\cdot\nabla = \left(u\hat{i} + v\hat{j} + w\hat{k}\right)\cdot\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \qquad \text{--- (4)}$$

Using equation (4), the equation (3) may be re-written as

$$a = \frac{dq}{dt} = \frac{Dq}{Dt} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)q + \frac{\partial q}{\partial t} = (q\cdot\nabla)q + \frac{\partial q}{\partial t} \qquad \text{----- (5)}$$

Which shows that the acceleration a of a fluid particle of fixed identity can be expressed as the material derivative of the velocity vector q.

Continuity equation or conservation of mass:

The law of conservation of mass states that fluid mass can be neither created nor destroyed. The equation of continuity aims at expressing the law of conservation of mass in a mathematical form. Thus, in continuous motion, the equation of continuity expresses the fact that the increase in the mass of the fluid within any closed surface drawn in the fluid in any time must be equal to the excess of the mass that flows in over the mass that flows out.

Let us consider a fixed closed surface S enclosing a volume V in the region occupied by a moving fluid. Let $\hat{n}$ be a unit outward normal vector drawn on the surface element δS, where fluid velocity is q and fluid density is ρ. Invert normal velocity is $-\hat{n}\cdot q$.

Thus,

Mass of the fluid entering across the surface δS in unit time i.e. Rate of mass flow across δS is

$$= \rho(-\hat{n}\cdot q)\delta S$$

Therefore,

Total mass of the fluid entering across the surface S in unit time i.e. Total rate of mass flow across S

$$= -\int_S \rho(\hat{n}\cdot q)\delta S = -\int_V \nabla\cdot(\rho q)dV \qquad \text{(By Gauss divergence theorem)} \qquad \text{------ (1)}$$

Since, the mass of the fluid within the volume V is $\int_V \rho\, dV$. Therefore, rate of increase (generation) of the fluid within the volume V i.e. Total rate of increase of mass of fluid within S is

$$\frac{\partial}{\partial t}\int_V \rho\, dV = \int_V \frac{\partial \rho}{\partial t} dV \qquad \text{----- (2)}$$

Suppose that the region V of the fluid contains neither sources nor sinks (i.e. there are no inlets or outlets through which fluid can enter or leave the region). Then by the law of conservation of the fluid mass, the rate of increase of the mass of fluid within V must be equal to the total rate of mass flowing into V. Hence from equation (1) and (2), we have

$$\int_V \frac{\partial \rho}{\partial t} dV = -\int_V \nabla\cdot(\rho q) dV$$

$$\Rightarrow \int_V \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho q)\right] dV = 0$$

Since, S is arbitrary so V is also arbitrary. Therefore, integral vanishes and we get

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho q) = 0 \qquad \text{----- (3)}$$

Equation (3) is called the Eulerian equation of continuity or the equation of conservation of mass and it holds at all points of fluid free from sources and sinks.

Special Cases:

1. Since $\nabla\cdot(\rho q) = q\cdot\nabla\rho + \rho\cdot\nabla q$, other forms of equation (3) are

$$\frac{\partial \rho}{\partial t} + q\cdot\nabla\rho + \rho\cdot\nabla q = 0$$

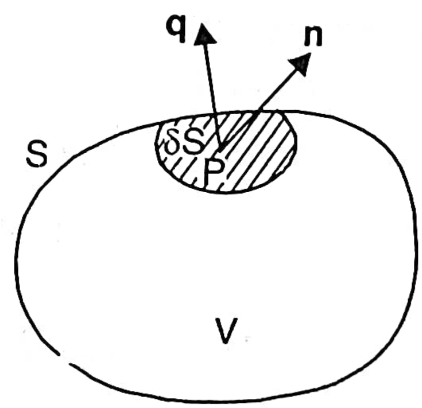
$$\left(\frac{\partial}{\partial t} + q\cdot\nabla\right)\rho + \rho\cdot\nabla q = 0$$

$$\frac{d\rho}{dt} + \rho\cdot\nabla q = 0$$

$$\frac{D\rho}{Dt} + \rho\cdot\nabla q = 0 \qquad \text{----- (4)}$$

And

$$\frac{D\log\rho}{Dt} + \nabla\cdot q = 0 \qquad \text{------ (5)}$$



2. For an incompressible and heterogeneous fluid the density of any fluid particle is invariable with time so that $\frac{D\rho}{Dt} = 0$. Then equation (4) gives

$$\nabla q = 0 \Rightarrow div\, q = 0$$

Or

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{if } q = u\hat{i} + v\hat{j} + w\hat{k}$$

3. For an incompressible and heterogeneous fluid, the density ρ is constant and hence $\frac{\partial \rho}{\partial t} = 0$. Then equation (3) gives,

$$\nabla\cdot(\rho q) = 0 \Rightarrow \nabla\cdot q = 0$$

Or

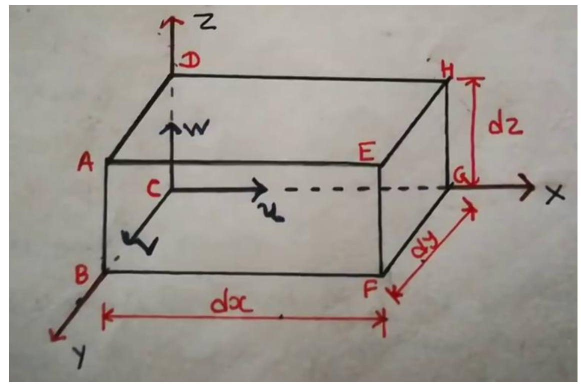
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \qquad \text{As } \rho \text{ is constant.}$$

Continuity Equation in fluid dynamics in three dimensions (in Cartesian Coordinates):

The basic continuity equation is an equation which describes the change of an intensive property. An intensive property is something which is independent of the amount of material you have. For instance, temperature would be an intensive property; heat would be the corresponding extensive property.

The equation based on the principle of conservation of mass is called continuity equation. Let us consider a fluid element of length dx, dy and dz in the direction of x, y and z. Let u, v u, and w are the inlet velocity component in x, y and z direction respectively.

Now, mass of the fluid entering the face ABCD (inflow) per second



$$= \frac{m}{S}$$
$$= \frac{\rho V}{S}$$
$$= \frac{\rho \times Area \times Length}{S}$$
$$= \frac{\rho \times A \times l}{S}$$
$$= \rho \times A \times v$$
$$= \rho \times Area\ of\ \ ABCD \times Velocity in\ x - direction$$
$$= \rho \times u \times dydz$$

Mass of fluid leaving the face EFGH (outflow) per second is

$$= \rho\, u\ dydz + \frac{\partial}{\partial x}(\rho\, u\ dydz)dx$$

Rate of increase in mass in x-direction or net mass of fluid remained in the element per unit time

= Mass through ABCD – Mass through EFGH per second

$$= \ \rho\, u\ dydz - \left\{\rho\, u\ dydz + \frac{\partial}{\partial x}(\rho\, u\ dydz)dx\right\}$$

$$= -\frac{\partial}{\partial x}(\rho\, u\ )dxdydz$$

Similarly,

Rate of increase in mass in y-direction $= -\frac{\partial}{\partial y}(\rho\, v\ )dxdydz$

And Rate of increase in mass in z-direction $= -\frac{\partial}{\partial z}(\rho\, w\ )dxdydz$

Therefore,

Total rate of increase in mass or net gain of mass i.e. total net mass of fluid remained in the element per unit time

$$= -\left[\frac{\partial}{\partial x}(\rho\, u\ ) + \frac{\partial}{\partial y}(\rho\, v\ ) + \frac{\partial}{\partial z}(\rho\, w\ )\right]dxdydz \qquad \text{---- (1)}$$

By the law of conservation of mass, there is no accumulation of mass i.e. mass is neither created nor destroyed in the fluid element. So, net increase of mass per unit time in the fluid element must be equal to the rate of increase of mass of fluid in the element.

Mass of fluid in element is $= \rho\ dxdydz$ $\left[\because \rho = \frac{m}{V} \Rightarrow m = \rho V\right]$

Its rate of increase with time $= \frac{\partial}{\partial t}(\rho\ dxdydz)$

$$= \frac{\partial \rho}{\partial t} dxdydz \qquad \text{---- (2)}$$

Equating equation (1) and (2), we get

$$-\left[\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)\right]dxdydz=\frac{\partial \rho}{\partial t}dxdydz$$

$$\Rightarrow \frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0 \qquad \text{----- (3)}$$

For steady flow (i.e. density does not change with time), $\frac{\partial \rho}{\partial t}=0$ and hence above equation becomes

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0$$

If the fluid is incompressible, then ρ is constant and the above equation becomes as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

This is the continuity equation in three-dimension.

For a 2D flow, the component $w=0$

$$\therefore \quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

Poiseuille's equation for flow of a liquid through a pipe:

Let us consider a liquid flowing trough horizontal tube of small diameter. The liquid can supposed to be composed of a number of co-axial cylindrical layers of varying radius whose axis coincides with the axis of the tube. The cylindrical layer in contact with the sides of the tube is permanently at rest due to the force of adhesion while that moving along the axis of the tube moves with maximum velocity. Thus, there exists a velocity gradient between different layers. A cross-sectional view of the velocity distribution of different layers is shown in fig.1,

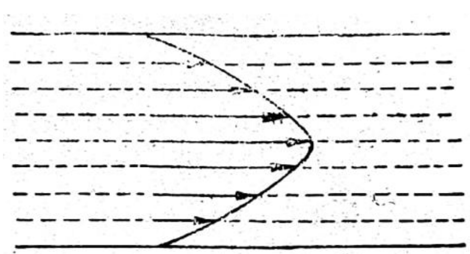


Fig.1: Flow of liquid through a tube

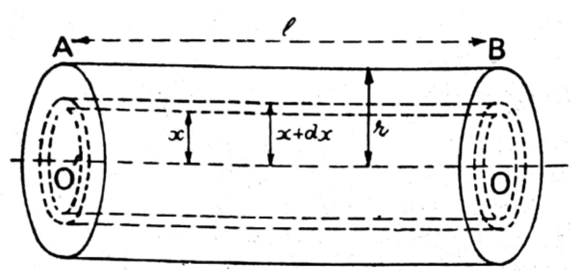


Fig.2 (a)

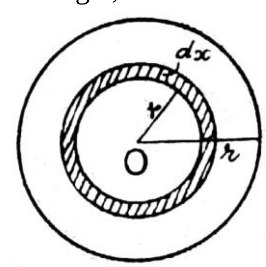


Fig.2 (b)

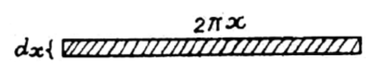


Fig.2 (c)

Fig.2: Cylindrical element inside a tube

Poiseuille's formula is the formula which tells us about the volume of liquid flowing per second across any cross-section of tube.

Let us consider a tube AB of length l and radius r held horizontally as shown in fig.2 (a). The liquid of co-efficient of viscosity η flows through the tube from A to B. Consider an elementary cylindrical layer of the liquid having internal radius x and thickness $x+dx$. The velocity of layer on the inside of this elementary cylindrical layer is slightly greater than that of outside one. Let their velocities be v and $v-dv$ respectively.

Due to the property of viscosity, the upper layer exerts back-ward drag F upon the lower layer and it is given by (According to Newton law of viscous flow, viscous force F acting tangentially on the layer of the liquid, opposite to the direction of flow is given by)

$$F=-\eta A\frac{dv}{dx}$$

Negative sign is due to the reason that if x increases, v decreases i.e. dv and dx posses opposite signs. Here, Area of cross-section, $A=2\pi xl$

$$\therefore \quad F = -\eta \times 2\pi x l \times \frac{dv}{dx} \qquad \text{-------- (1)}$$

Let P_1 and P_2 be the pressures on the two sides of tubes.

Force due to pressure P_1 (from left to right), $F_1 = \pi x^2 P_1$

Force due to pressure P_2 (from right to left), $F_2 = \pi x^2 P_2$

If the liquid flows from left to right, it will flow only if $P_1 > P_2$.

Therefore,

Net force on the liquid (from left to right),

$$F' = F_1 - F_2 = \pi x^2 (P_1 - P_2) = \pi x^2 P \qquad \text{---- (2)}$$

Where, $P = P_1 - P_2$ is the difference of pressure on the two ends of the tube.

In equilibrium, when the liquid flows in steady flow

$$F = F'$$

$$\Rightarrow -\eta \times 2\pi x l \times \frac{dv}{dx} = \pi x^2 P$$

$$\Rightarrow dv = -\frac{P}{2\eta l} x dx$$

Integrating, we get

$$\int dv = -\frac{P}{2\eta l} \int x dx$$

$$\Rightarrow v = -\frac{P}{2\eta l} \frac{x^2}{2} + c$$

$$\Rightarrow v = -\frac{P}{4\eta l} x^2 + c \qquad \text{----- (3)}$$

Here, c is the constant of integration. Since the layer of liquid in contact with the sides of the tube is at rest i.e. at $x = r, \quad v = 0$. Therefore from equation (3), we get

$$\Rightarrow 0 = -\frac{P}{4\eta l} r^2 + c$$

$$\Rightarrow c = \frac{P}{4\eta l} r^2$$

Substituting the value of c in equation (3), we get

$$v = -\frac{P}{4\eta l} x^2 + \frac{P}{4\eta l} r^2$$

$$\Rightarrow v = \frac{P}{4\eta l}\left(r^2 - x^2\right) \qquad \text{------ (4)}$$

Equation (4) gives the velocity of the liquid flowing through the tube.

A cross-sectional view of the flow of liquid is shown in fig.2 (b). Shaded region gives the face area of the cylindrical layer. Imagine the face area to be cut and spread in the form of rectangle as shown in fig.2(c)

Now, the volume of liquid that flows out per second through the cylindrical shell or Rate of flow of liquid through the tube is

$$dV = Area\ of\ cross-section\ of\ the\ shell \times Velocity\ of\ flow\ of\ liquid\ through\ this\ shell\ v$$

$$\Rightarrow dV = 2\pi x dx \times \frac{P}{4\eta l}\left(r^2 - x^2\right)$$

$$\Rightarrow dV = \frac{\pi P}{2\eta l}\left(r^2 x - x^3\right) dx$$

Therefore, the total volume of liquid flowing per second across any cross-section of the tube is

$$V = \int dV = \frac{\pi P}{2\eta l}\int_0^r \left(r^2 x - x^3\right)dx$$

$$\Rightarrow V = \frac{\pi P}{2\eta l}\left[r^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^r$$

$$\Rightarrow V = \frac{\pi P}{2\eta l}\left[\left(r^2\frac{r^2}{2} - \frac{r^4}{4}\right) - (0-0)\right]$$

$$\Rightarrow V = \frac{\pi P}{2\eta l}r^4\left(\frac{1}{2} - \frac{1}{4}\right)$$

$$\Rightarrow V = \frac{\pi P}{2\eta l}r^4\left(\frac{2-1}{4}\right)$$

$$\Rightarrow V = \frac{\pi P}{8\eta l}r^4 \qquad \text{----- (5)}$$

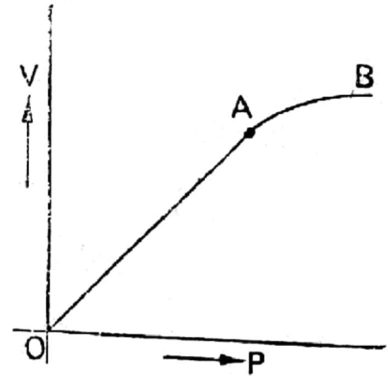


Fig.3

Equation (5) is called the Poiseuille's equation or Poiseuille's formula. This relation holds good only for tubes of smaller diameter and for stream line flow.

Figure.3 shows the variation of rate of flow (V) i.e. volume of liquid flowing per second with the pressure difference P between the two ends. For smaller P, region OA corresponds to the state when the velocity of flow of liquid is less than the critical velocity v_c so that the flow is streamlined. In this region, $V \propto P$ as stated by Poiseuille's formula. On increasing the pressure beyond A, the velocity of liquid increases beyond v_c, thus making the flow turbulent. Now V is not proportional to P but it is proportional to $\sqrt{P}$. So Poiseuille's formula does not hold good in region AB.

Again, if the height of the liquid above the axis of the tube is 'h' then Pressure

$$P = h\rho g$$

$$\therefore \quad V = \frac{\pi P}{8\eta l}r^4 = \frac{\pi r^4 h\rho g}{8\eta l}$$

Limitation of Poiseuille's formula:

1. The formula applies only to streamline flow through the tube. The flow is streamline when the velocity of flow is less than critical velocity. Since critical velocity of a liquid is inversely proportional to the radius of the tube, this flow will tend to become turbulent in case of tubes of wide bore. **Thus, Poiseuille's formula holds good for narrow tube only.**
2. The formula breaks down if the liquid layers in contact with the walls are not stationary. For it the pressure difference across the capillary should be kept low so that liquid flows very slowly through the tube.
3. Poiseuille's formula holds good only so long as the tube is horizontal and escaping fluid has negligible kinetic energy.
4. Poiseuille's formula is not valid for gas.