Special Theory of Relativity

The special Theory of Relativity (STR) was developed by Albert Einstein in 1905 and in 1915 proposed the general theory of relativity. The theory of special relativity explains how space and time are linked for objects that are moving at a consistent speed. The STR deals with the problems in which one frame of reference moves with a constant linear velocity relative to another frame of reference. The general theory of relativity (GTR) deals with problems in which one frame of reference is accelerated with respect to another frame of reference.

Postulates of STR:

1. All the laws of physics are applicable (*or same*) in all inertial frames of reference. (Or the laws of physics, including Maxwell theory of electromagnetism are invariant (never changing, remains unchanged) in all inertial frames of reference)
2. The velocity of light in free space (vacuum) is constant i.e. $3\times10^8 m/s$ and is independent of motion of observer or source of light in any frame of reference. (Or the speed of light in a vacuum is the same for all observers, regardless of the motion of the light source or observer.)

Frame of reference:

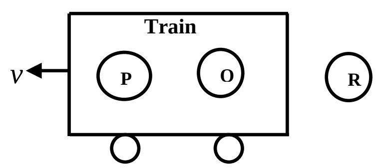
A frame of reference is a system of geometric axes in relation to which measurement of size, position or motion can be made.

Inertial frame of reference:

Those frame of reference in which Newton's law of motion hold are called Inertial Frame of Reference i.e. a reference frame in which an object remains either at rest or at a constant velocity unless another force acts upon it is called inertial frame of reference.. Example: A person inside a stationary care, moving train with constant velocity.

Non-inertial frame of reference:

A non-inertial frame is any frame where some external force works upon it. Exm: An accelerating car pushes us back into the seat.



Let us consider a train moving with a velocity v in negative x-direction and two person P & Q are sitting inside the train and this persons are stationary and the 3rd person are standing outside the train. So in this scenario, for person P, person Q is at rest but for person R, person Q is travelling in negative x-direction with velocity v i.e. in the direction of the train i.e. inertial frame is also relative.

So which is inertial frame?

Actually, first we need to assume an inertial frame for us before going further. So a frame is inertial frame if it is at rest/moving with constant velocity with respect to our assumed inertial frame. So here, P is inertial frame for Q and vice versa but P is non inertial frame for R.

Lorentz Transformation:

The Lorentz transformation is the transformation between two inertial reference frames when one is moving with a constant velocity with respect to the other.

Let us consider two frame of reference S and S'. S is fixed and S' is moving with a constant speed v along the direction of the x-axis. Initially both the frames have the same origin of coordinates. After a time t the frame of reference S' has moved a distance $OO' = vt$. Let two observers O and O' observe any event P from system S and S'. Let for the point P in space, the coordinates are (x, y, z) with reference to the frame S and (x', y', z') with reference to the frame S'.

According to the Lorentz the measurement in the x-direction made in frame S should be proportional to that made in S', i.e., the equation between x and x' must be linear and be of the form

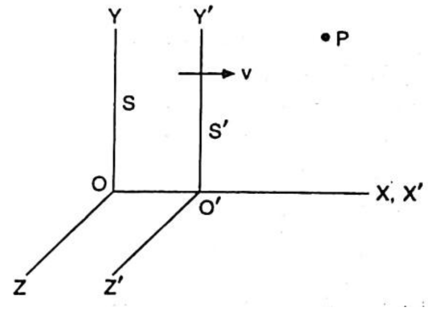
$$x' = k(x - vt) \qquad \text{------ (1)}$$

Where, k is a constant proportionality that is independent of x and t and it is called Lorentz factor.
Since, according to STR postulate-I, the laws of physics should have same form in both frames of reference S and S', therefore the corresponding equation for x in terms of x' and t' will be of the same form as equation (1) except that v is replaced by $-v$, because S may be assumed to move relative to S' with velocity $-v$.

$$\therefore \quad x = k(x' + vt') \qquad \text{----- (2)}$$

Since the relative motion of the systems S and S' is only along X, X' axis, the coordinate perpendicular to the motion should remain unchanged, hence

$$y = y' \text{ and } z = z' \qquad \text{---- (3)}$$



The time coordinates t and t' are however not equal.
Substituting the values of x' from equation (1) in equation (2), we get

$$\therefore \quad x = k[k(x - vt) + vt']$$

$$\Rightarrow x = k^2(x - vt) + kvt'$$

$$\Rightarrow kvt' = x - k^2x + k^2vt$$

$$\Rightarrow kvt' = x(1 - k^2) + k^2vt$$

$$\Rightarrow t' = \frac{x(1-k^2)}{kv} + \frac{k^2vt}{kv}$$

$$\Rightarrow t' = kt + x\frac{(1-k^2)}{kv} \qquad \text{---- (4)}$$

(*The value of constant k can be evaluated from postulate-II*). For it, let a signal of light be emitted from the common origin of S and S' at time $t = t' = 0$. The signal travels with speed c, which is the same for both frames (postulate-II). After some time, the position of the signal, as seen from S and S' is given by

$$x = ct \qquad \text{--- (5)}$$

And

$$x' = ct' \qquad \text{--- (6)}$$

Substituting the values of x' and t' from equation (1) and (4) in equation (6), we get

$$x' = ct'$$

$$\Rightarrow k(x - vt) = c\left[kt + x\frac{(1-k^2)}{kv}\right]$$

$$\Rightarrow kx - cx\frac{(1-k^2)}{kv} = ckt + kvt$$

$$\Rightarrow x\left[k - c\frac{(1-k^2)}{kv}\right] = ckt\left(1 + \frac{v}{c}\right)$$

$$\Rightarrow x = \frac{ckt\left(1+\frac{v}{c}\right)}{\left[k - \frac{(1-k^2)}{k}\frac{c}{v}\right]}$$

$$\Rightarrow x = \frac{ckt\left(1+\frac{v}{c}\right)}{k\left[1 - \frac{(1-k^2)}{k^2}\frac{c}{v}\right]}$$

$$\Rightarrow x = \frac{ct\left(1+\frac{v}{c}\right)}{1 - \left(\frac{1}{k^2} - 1\right)\frac{c}{v}}$$

$$\Rightarrow ct = \frac{ct\left(1+\frac{v}{c}\right)}{1-\left(\frac{1}{k^2}-1\right)\frac{c}{v}} \qquad \text{(Using equation 5)}$$

$$\Rightarrow \frac{1}{k^2}\frac{c}{v} = \frac{c}{v} - \frac{v}{c}$$

$$\Rightarrow \frac{1}{k^2}\frac{c}{v} = \frac{c^2-v^2}{vc}$$

$$\Rightarrow \frac{1}{k^2} = \frac{c^2-v^2}{vc}\times\frac{v}{c}$$

$$\Rightarrow \frac{1}{k^2} = \frac{c^2-v^2}{c^2}$$

$$\Rightarrow \frac{1}{k^2} = 1-\frac{v^2}{c^2}$$

$$\Rightarrow k = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{--- (7)}$$

Substituting this value of k in equation (1), we get

$$x' = k(x-vt)$$

$$\Rightarrow x' = \frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{---- (8)}$$

Again, substituting the value of k in equation (4), we get

$$\Rightarrow t' = kt + x\frac{\left(1-k^2\right)}{kv}$$

$$\Rightarrow t' = k\left[t + \frac{\left(1-k^2\right)}{k^2}\frac{x}{v}\right]$$

$$\Rightarrow t' = k\left[t + \left(\frac{1}{k^2}-1\right)\frac{x}{v}\right]$$

$$\Rightarrow t' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left[t + \left(1-\frac{v^2}{c^2}-1\right)\frac{x}{v}\right]$$

$$\Rightarrow t' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left[t - \frac{v^2}{c^2}\frac{x}{v}\right]$$

$$\Rightarrow t' = \frac{t-\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{--- (9)}$$

Equations (3), (8) and (9) are known as Lorentz transformation equations from frame of reference S to S'. They are reproduced below in their complete form

$$x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$y' = y$$

$$z' = z \qquad \text{---- (10)}$$

And $$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

The inverse transformations from system S' to system S can be obtained by replacing v by $-v$ and interchanging primed and unprimed coordinates. Thus inverse Lorentz transformation equations are

$$x = \frac{x' + vt'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$y = y'$$

$$z = z' \qquad \text{--- (11)}$$

And $$t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

A significant conclusion from Lorentz transformation is that nothing can move with a velocity greater than the velocity of light i.e. v should be less than c. If $v > c$, $\sqrt{1-\frac{v^2}{c^2}}$ would become imaginary making place and time coordinates also imaginary, which is impossible.

Q: Show that for low values of v, Lorentz transformations approach to Galilean transformation.

Ans; We know that, Lorentz transformation equations are

$$x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}},\ y' = y,\ z' = z \quad \text{and} \quad t' = \frac{t - \frac{xv}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

For low velocity (i.e. in Newtonian or Classical mechanics), $v << c$ so that $\frac{v}{c} \rightarrow 0$ and then Lorentz transformations reduce to

$$x' = x - vt,\ y' = y,\ z' = z \quad \text{and} \quad t' = t$$

Which are Galilean transformation equations.

Q: A light pulse is emitted at the origin of a frame of reference S' at time $t' = 0$. Its distance is x' from the origin, after time t' is given by $x'^2 = c^2 t'^2$. Use Lorentz transformations to transform this equation to an equation in x and t and show that this is $x^2 = c^2 t^2$. Discuss the implications of this result.

Ans: Given that, $x'^2 = c^2 t'^2$ ---- (1)

Since, Lorentz transformation equations are

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}},\ y' = y,\ z' = z \quad \text{and} \quad t' = \frac{t - \frac{xv}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{--- (2)}$$

Substituting the values of x' and t' from equation (2) in equation (1), we get

$$x'^2 = c^2 t'^2$$

$$\Rightarrow \frac{(x - vt)^2}{1 - \frac{v^2}{c^2}} = c^2 \frac{\left(t - \frac{xv}{c^2}\right)^2}{1 - \frac{v^2}{c^2}}$$

$$\Rightarrow (x - vt)^2 = c^2 \left(t - \frac{xv}{c^2}\right)^2$$

$$\Rightarrow x^2 + v^2 t^2 - 2xvt = c^2 \left(t^2 + \frac{x^2 v^2}{c^4} - \frac{2txv}{c^2}\right)$$

$$\Rightarrow x^2 + v^2 t^2 - 2xvt = c^2 t^2 + \frac{x^2 v^2}{c^2} - 2xvt$$

$$\Rightarrow x^2 - c^2 t^2 + v^2 t^2 - \frac{x^2 v^2}{c^2} = 0$$

$$\Rightarrow \left(x^2 - c^2 t^2\right) - \frac{v^2}{c^2}\left(x^2 - c^2 t^2\right) = 0$$

$$\Rightarrow \left(x^2 - c^2 t^2\right)\left(1 - \frac{v^2}{c^2}\right) = 0$$

Now, $\left(1 - \frac{v^2}{c^2}\right) \neq 0$, because $v \neq c$

$$\therefore \left(x^2 - c^2 t^2\right) = 0$$

$$\Rightarrow x^2 = c^2 t^2$$

This result indicates that the velocity of light c is the same in all the systems i.e., c is an absolute constant and independent of the frame of reference.

Minkowski's Space: (Four Dimensional Space–time Continuum)

Hermann Minkowski pointed out that the external world is not formed of ordinary three dimensional space coordinates x, y & z, known as Euclidean space, but it is made up of four dimensional space– time continuums x, y, z & $time$, known as Minkowski space or world space or four space i.e. Minkowski space is a combination three dimensional Euclidean space and time.

Any event is denoted by its position in space and the time of its occurrence. Thus we live in a four dimensional world i.e. in a world of events rather than the world of points. This is called a space-time continuum i.e. space and time cannot be separated. To give the time coordinate, the same dimension of length as x, y & z coordinate, the time coordinate is multiplied by the velocity of light c and is chosen as an imaginary quantity i.e. ict, where $i = \sqrt{-1}$.

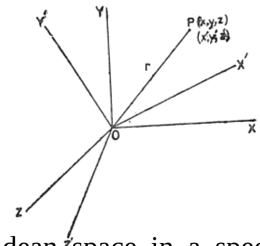
Let us consider two coordinate systems x, y, z and x', y', z' with common origin at O. According to three dimensional space, the position of P with respect to the two coordinate systems is given by

$$r^2 = x^2 + y^2 + z^2$$

And also, $$r^2 = x'^2 + y'^2 + z'^2$$

Whatever may be the values of x, y & z, for coordinate axes with common origin, the value of r^2 remains constant.

According to Minkowski, a fourth dimension $\tau = ict$ is introduced.



$$\therefore \quad r^2 = x^2 + y^2 + z^2 + \tau^2$$

$$\Rightarrow r^2 = x^2 + y^2 + z^2 - c^2t^2 \qquad \text{---- (1)}$$

And

$$r^2 = x'^2 + y'^2 + z'^2 + \tau'^2$$

$$\Rightarrow r^2 = x'^2 + y'^2 + z'^2 - c^2t'^2 \qquad \text{---- (2)}$$

Equation (1) and (2) indicate that the space time continuum differs from the Euclidean space in a specific way. Here τ^2 or τ'^2 has a negative sign and r^2 can be zero as $x^2 + y^2 + z^2$ will be a positive value. This, however is not possible in a three dimensional space.

Space–Time interval: (The invariant interval)

The space time interval has the same value in all reference frames i.e. space time interval is invariant quantity i.e.

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2 \qquad \text{--- (1)}$$

Where (x, y, z) and (x', y', z') are the co-ordinates of the system event observed by two observers in system S and S' while S' is moving with a velocity v relative to S.

We know that Lorentz Transformation equations are

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \; y' = y, \; z' = z \quad \text{and} \quad t' = \frac{t - \frac{xv}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \text{--- (2)}$$

Now, R.H.S of equation (1),

$$x'^2 + y'^2 + z'^2 - c^2t'^2$$

$$= \frac{(x - vt)^2}{1 - \frac{v^2}{c^2}} + y^2 + z^2 - c^2 \frac{\left(t - \frac{xv}{c^2}\right)^2}{1 - \frac{v^2}{c^2}} \qquad \text{(Using Lorentz Transformation equations)}$$

$$= \frac{x^2 + v^2t^2 - 2xvt - c^2\left(t^2 + \frac{x^2v^2}{c^4} - \frac{2txv}{c^2}\right)}{1 - \frac{v^2}{c^2}} + y^2 + z^2$$

$$= \frac{x^2 + v^2t^2 - 2xvt - c^2t^2 - \frac{x^2v^2}{c^2} + 2xvt}{1 - \frac{v^2}{c^2}} + y^2 + z^2$$

$$= \frac{x^2 - \frac{x^2v^2}{c^2} + v^2t^2 - c^2t^2}{1 - \frac{v^2}{c^2}} + y^2 + z^2$$

$$= \frac{x^2\left(1 - \frac{v^2}{c^2}\right) - c^2t^2\left(1 - \frac{v^2}{c^2}\right)}{1 - \frac{v^2}{c^2}} + y^2 + z^2$$

$$= \frac{\left(1-\frac{v^2}{c^2}\right)\left(x^2-c^2t^2\right)}{1-\frac{v^2}{c^2}} + y^2 + z^2$$

$$= \left(x^2-c^2t^2\right) + y^2 + z^2$$

$$= x^2 + y^2 + z^2 - c^2t^2 \text{ (L.H.S)}$$

Thus we have proved that

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2$$

That is, the expression $x^2 + y^2 + z^2 - c^2t^2$ is invariant under Lorentz transformations.

(Same can be proved by using the inverse Lorentz Transformation equations as

We know that Lorentz Transformation equations are

$$x = \frac{x' + vt'}{\sqrt{1-\frac{v^2}{c^2}}},\ y = y',\ z = z' \quad \text{and} \quad t = \frac{t' + \frac{xv}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Now, L.H.S,

$$x^2 + y^2 + z^2 - c^2t^2 \text{ (Home work)}$$

Q: show that space time interval is invariant under Lorentz Transformation.

OR: show that $x^2 + y^2 + z^2 - c^2t^2$ is invariant under L.T.

World Point:

A physical event in Minkowski space is described by a point with coordinates x, y, z, ict which is called world point.

World line:

The path of a particle in the Minkowski space corresponds to a line called world line.

Light Cone:

A light cone in a space time diagram is the cone formed by the light rays as they originate or end up at the origin.

We can interpret the light cone as a region in space time which we can influence if we are present at the origin which is the joining point of those two cones, past light cone and future light cone. Sitting at the origin, we can never send any signal to a point which is outside the light cone in the future because nothing can ever travel at a speed greater than the speed of light. All points in our future will always lie inside our future light cone. Similarly, only the past events which lie inside the past light cone have no way of reaching us now, due to the finite speed of light.

Space – Time Diagram: (Light Cone)

To represent the motion of a particle graphically, we plot the position versus time graph. Velocity can be obtained from the slop of the position time graph. A particle at rest is represented by vertical line.

A space time diagram is nothing more than a graph showing the position of objects as a function of time. For simpler explanation Minkowski took one space dimension (x) over horizontal x-axis and time dimension (t) over vertical y-axis. So x-axis would represent space and y-axis would represent time. Because the speed of light is special, space-time diagram are often drawn in units of seconds and light and light seconds or years and light years, so a unit slop (45^0 angle) corresponds to the speed of light.

Photon travelling with speed of light is described by a 45^0 line. Rocket going at some intermediate speed follows a line of slop $\frac{c}{v}=\frac{1}{\beta}$. Such plots are called Minkowski diagram. The trajectory of a particle on Minkowski diagram is called World Line.

Time Dilation:

(*Let's talk about clock: Think of a clock as a heartbeat. If you are in running, your heart beats faster and if you walk, your heart beat slower. But, time works the other way around. The slower you move, the faster a clock ticks and the faster you move, the slower a clock ticks.*

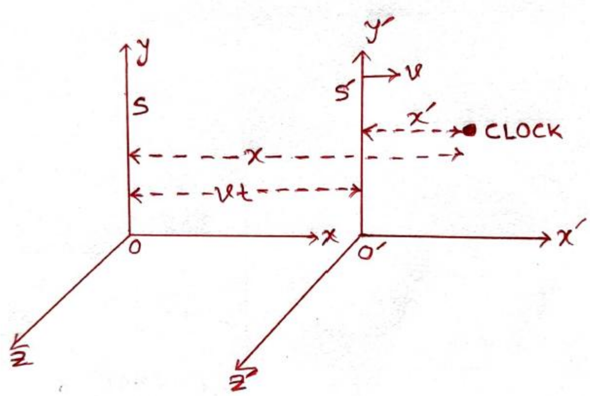
Definition: Time slows down at very very fast speed. This phenomenon is known as Time Dilation.)

Let us consider two frames of reference S and S'. Let S' be moving with a velocity v with respect to S along positive direction of x-axis. Let a clock be fixed at the point x' in the moving frame S'. Let this clock give a signal at time t_1 in system S and suppose that t_1' is the time measured by the observer in S' corresponding to the time t_1. Then from inverse transformation equations, we have

$$t_1=\frac{t_1'+\frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{--- (1)}$$

If this clock again gives a signal at time t_2, then corresponding time t_2' in frame S' is given by

$$t_2=\frac{t_2'+\frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{--- (2)}$$



From equation (1) and (2), we have

$$t_2-t_1=\frac{t_2'+\frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}-\frac{t_1'+\frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow t_2-t_1=\frac{t_2'+\frac{vx'}{c^2}-t_1'-\frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow t_2-t_1=\frac{t_2'-t_1'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow \Delta t=\frac{\Delta t'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow \Delta t=\Delta t'\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

$$\Rightarrow \Delta t=\Delta t'\left(1+\frac{v^2}{2c^2}\right) \qquad \text{--- (3)}$$

i.e. $\Delta t>\Delta t'$

It is clear from expression (3) that time interval $\Delta t'$ appears to the moving observer to be dilated or to an observer in stationary frame S, the time interval appears to be lengthened by a factor, $\sqrt{1-\frac{v^2}{c^2}}$, where v is the velocity of the moving observer. It implies that the "***A moving clock always appears to go slow***". This effect is called time dilation. This effect is reciprocal i.e., when the observer in S' looks at the clock in S, it appears to be running slow as compared to his own. Thus ***"Every clock appears to run at its fastest rate when it is at rest relative to the observer and appears to be slowed down by a factor*** $\sqrt{1-\frac{v^2}{c^2}}$ ***when it moves with a velocity*** v ***relative to the observer".*** The time interval Δt measured in a frame in which the clock is at rest is called the **proper time interval**. Time interval $\Delta t'$ shown by any other clock that is in relative motion with respect to observer is called **local time interval or improper time interval**.

Time Dilation: (OR)

Let us consider two frames of reference S and S'. Let S' be moving with a velocity v with respect to S along positive direction of x-axis. Let a clock be fixed at the point x from frame S (i.e. let a clock be placed in reference system S which is at rest). It shows time of event in both the frames. Time interval shown by a clock that is at rest relative to an observer is called the proper time interval. Time interval shown by any other clock that is in relative motion with respect to observer is called local time interval or improper time interval. Let this clock give a signal at time t_1 in system S and suppose that t_1' is the time measured by the observer in S' corresponding to the time t_1. Then from transformation equations, we have

$$t_1' = \frac{t_1 - \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{--- (1)}$$

If this clock again gives a signal at time t_2 , then corresponding time t_2' in frame S' is given by

$$t_2' = \frac{t_2 - \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{--- (2)}$$

From equation (1) and (2), we have

$$t_2' - t_1' = \frac{t_2 - \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{t_1 - \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow t_2' - t_1' = \frac{t_2 - \frac{vx}{c^2} - t_1 + \frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow t_2' - t_1' = \frac{t_2 - t_1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow \Delta t' = \frac{\Delta t}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\Rightarrow \Delta t' = \Delta t \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

$$\Rightarrow \Delta t' = \Delta t\left(1+\frac{v^2}{2c^2}\right) \qquad \text{--- (3)}$$

i.e. $\Delta t' > \Delta t$

It is clear from expression (3) that time interval Δt appears to the moving observer to be dilated or lengthened by a factor, $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, where v is the velocity of the moving observer.

Hence from above reasoning we may say "***A moving clock always appears to go slow***". Consequently to the observer in motion the clock at rest appears to be retarded by the factor $\sqrt{1-\frac{v^2}{c^2}}$. This is apparent relation of clocks. The above reasoning may be summarized as a rule. ***"Every clock appears to go at its faster rate when it is at rest relative to the observer. Its rate appears to go on slowing by the factor*** $\sqrt{1-\frac{v^2}{c^2}}$ ***as its velocity*** v ***relative to the observer goes on increasing"***.

1. When $v \ll c$, $\frac{v^2}{c^2}$ is negligible then $\Delta t' = \Delta t$. This shows that the time interval remains the same as that when it is at rest.
2. When v is comparable to c then $\Delta t' > \Delta t$. This shows that time interval recorded in a moving frame S' is greater than the time interval recorded by the clock at frame S when it is at rest.
3. when $v > c$, $\frac{v^2}{c^2} > 1$, hence $\Delta t' = \infty$ or imaginary. This shows that no material particle can ever attain the velocity of light.

Twin Paradox:

Let there are two twins A and B, each 20 years of age. A remaining at rest at the origin O of a frame S, whereas B travels within a space ship S' with a velocity $\frac{\sqrt{3}}{2}c$ for 1 year (time of system S') along positive direction ox X-axis. Then he returns with same speed and reaches the starting point after the end of another year of his time.

Then according to B, the age of B is 20 + 2 = 22 years, but according to A, the age of B is

$$20+\Delta t' = 20+\frac{\Delta t}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= 20+\frac{2}{\sqrt{1-\frac{\left(\frac{\sqrt{3}}{2}c\right)^2}{c^2}}}$$

$$= 20+\frac{2}{\sqrt{1-\frac{3}{4}}}$$

$$= 20+\frac{2}{\sqrt{1-0.75}}$$

$$= 20+\frac{2}{\sqrt{0.25}}$$

$$= 20+\frac{2}{0.5}$$

$$= 20+4$$

= 24 years

On the other hand, the age of A according to A would be 24 years, whereas the age of A according to B would be 22 years. There two statements are different. Will one of the two twins appear younger that the other. This is twin paradox in special theory of relativity.

According to special theory of relativity the resolution is that A will be 24 years old while B 22 years. That is B is younger that A. There is no symmetry between B and A, because B changes his frame with A remaining in same frame.

An interesting example of time dilation. Imagine that once a 40 years old scientist fell in love with his 16 years laboratory assistant. They want to marry ; but they feel that their marriage cannot be welcomed by the society due to the age difference. Scientist plans to marry using the principle of time dilation of relativity. He synchronises his clock with that of his assistant and goes to a long space journey in a spaceship moving with velocity 0.999 c. He returns back when his clock reads one year and reads the clock of his assistant. He finds that in here clock

$$\left[\Delta t' = \frac{\Delta t}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}} = \frac{1}{\left\{\sqrt{\left(1-\frac{0\cdot999\,c}{c}\right)^2}\right\}} = 22\cdot7 \text{ years}\right]$$

have passed. This means that now scientist is 41 years old and his assistant (16 + 22·7) = 38·7 years old. Their age difference barrier has now been overcome, so they marry reciting the limerick :

"There once was a lady called Bright,
Who could travel faster than light ;
She went out one day, in a relative way,
And came back the previous night."

Space like, Time like and Light like Intervals:

Let us consider two frames of reference S and S'. Let S' be moving with a velocity v with respect to S along positive direction of x-axis. Let initially both the frames have the same origin of coordinates.

Now, let us consider two events having coordinates (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) in system S, then the interval between the two events in system S

$$S_{12} = \left[c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2\right]^{\frac{1}{2}} \qquad \text{--- (1)}$$

Let in the system S' which is moving with uniform velocity v relative to system S, the coordinates of these events be (x_1', y_1', z_1', t_1') and (x_2', y_2', z_2', t_2'), then the interval between the two events in system S' is given by

$$S_{12}' = \left[c^2(t_2' - t_1')^2 - (x_2' - x_1')^2 - (y_2' - y_1')^2 - (z_2' - z_1')^2\right]^{\frac{1}{2}} \qquad \text{--- (2)} \quad \text{Since, } c \text{ in invariant.}$$

But according to Lorentz Transformations, we have

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}, \; y' = y, \; z' = z \quad \text{and} \quad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\therefore \; S_{12}'^2 = \left[c^2\left\{\frac{t_2 - \frac{vx_2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{t_1 - \frac{vx_1}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}\right\}^2 - \left\{\frac{x_2 - vt_2}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{x_1 - vt_1}{\sqrt{1 - \frac{v^2}{c^2}}}\right\}^2 - (y_2' - y_1')^2 - (z_2' - z_1')^2\right]$$

$$\Rightarrow \; S_{12}'^2 = \left[c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2\right] \quad \text{(After simplification)}$$

$$\Rightarrow \; S_{12}'^2 = S_{12}^2 \qquad \text{---- (1)}$$

That is the interval between two events in Minkowski space is independent of the frame of reference i.e., it is invariant. In other words, the interval between two events is invariant under transformation from one inertial system to another inertial system.

Thus, $\left[c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2\right] = \left[c^2(t_2' - t_1')^2 - (x_2' - x_1')^2 - (y_2' - y_1')^2 - (z_2' - z_1')^2\right]$

If the two events considered occur at the same point in the space S' i.e.

$$x_2' = x_1', \; y_2' = y_1' \text{ and } z_2' = z_1'$$

$$\therefore \; S_{12}^2 = c^2(t_2' - t_1')^2 \qquad \text{--- (2)}$$

But $(t_2' - t_1') > 0$, since we have considered that the second event occurs after the first.

$$\therefore \; c^2(t_2 - t_1)^2 > 0$$

So that, $\quad S_{12}^2 > 0 \qquad$ --- (3)

Hence, for S_{12}^2 to be greater than zero, there will be a system or which the interval between the two events is real. Real intervals are known as **"Time like intervals"**.

The condition for time like intervals is

$$c^2(t_2 - t_1)^2 > (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \qquad \text{--- (4)}$$

Thus we may say that if the interval between two events is "Time like interval"; then there exist a system in which the two events take place at one point.

Now consider that the two events take place at the same time in system S', then $t_2' = t_1'$

$$\therefore \; S_{12}^2 = c^2(t_2' - t_1')^2 - (x_2' - x_1')^2 - (y_2' - y_1')^2 - (z_2' - z_1')^2$$

$$\Rightarrow S_{12}^2 = -(x_2' - x_1')^2 - (y_2' - y_1')^2 - (z_2' - z_1')^2$$

$$\Rightarrow S_{12}^2 = -\left[(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2\right]$$

$$\Rightarrow S_{12}^2 = (-)ve \; quantity$$

$$i.e. \; S_{12}^2 < 0 \qquad \text{---- (5)}$$

Thus S_{12}^2 is $(-)ve$ or S_{12} is imaginary. Such intervals are known as **"Space like interval"**. Therefore, if the interval between the two events is imaginary, the intervals are said to be **"Space like interval"**. Thus if the interval between two events is space like interval, there exists a system of reference in which the events take place simultaneously. The condition for space like intervals is given by

$$c^2(t_2-t_1)^2 < (x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2 \qquad \text{---- (6)}$$

Now, if we assume that

$$\rho_{12}^2 = (x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2 - c^2(t_2-t_1)^2 = -S_{12}^2$$

Then interval between two events is time like; if either S_{12} is real or ρ_{12} is imaginary. The interval between two events is space like; if either ρ_{12} is real or S_{12} is imaginary.

Now, if S_{12} is zero then the interval is **singular or light like**. Thus the time like or space like intervals may be separated by

$$S_{12}^2 = 0$$

$$\Rightarrow \; c^2(t_2-t_1)^2-(x_2-x_1)^2-(y_2-y_1)^2-(z_2-z_1)^2 = 0$$

Which is known as **Null Cone**.

Now, let us

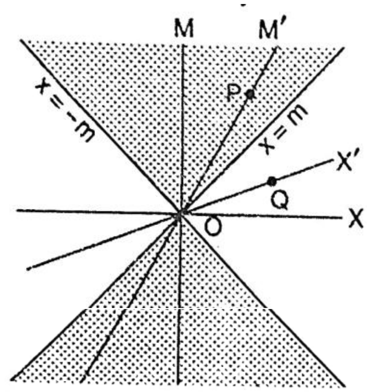


Fig. 5.8

Now let us represent space like and time like intervals in Minkowski space geometrically. Consider the shaded area of fig 5.8 bounded by world lines of light waves $x=m$ and $x=-m$. Through any point P we draw m'-axis through the origin O, *i.e.*, we can find an inertial frame S' in which events O and P occur at the same place ($x'=0$) and are separated only in time. From fig. 5.8 it is obvious that the event P occurs at a later instant than event O in system S', *i.e.*, even P follows event O in time when the event P is in the upper half of the shaded area. Hence the events in the upper half (region 1 fig. 5.9) are absolutely in the future relative to O and this region is called **absolute future**. If the event P

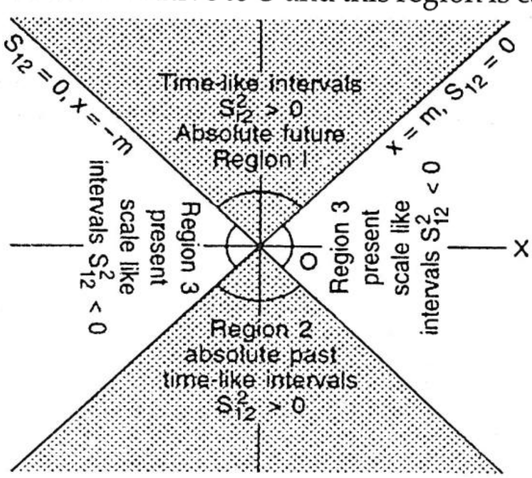


Fig. 5.9

lies in the lower half of the shaded area (region 2 fig. 5.9), then event P occurs earlier than O. Hence the events in the lower half are absolutely in the past relative to O and this region is called the **absolute past.** Thus in the shaded region there is a definite time order of events relative to O, because we can always find a frame in which O and P occur at the same place. Therefore a single clock will determine the time order of the event at this place.

Now Let us consider the unshaded region. Through any point Q we draw x'-axis from the origin, *i.e.*, we can find an inertial system S' in which the events O and Q occur at the same time and are separated only in space. There is always an inertial system in which events O and Q appear to be simultaneous for space-time points in the unshaded region (region 3 of fig 5.9) and this region is called the *present*. In other inertial frames O and Q are not simultaneous and there is no absolute time order of these events, but a relative time order, instead.

... that the events in the present are absolutely