# Small Amplitude Oscillations

In this category, we will discuss about the mechanics of the particles having small amplitude of oscillations. In case of mutually interacting particles, the motion of one particle in influenced by other particles and so the entire system develops a different mode of motion i.e. called 'normal mode of motion'.

## Stability Analysis:

By stability analysis we mean, finding equilibrium position and investigating whether the given equilibrium is stable or unstable.

Equilibrium criteria in one dimension:

If $V$ be the potential energy then the relation between force and potential energy is

$$F = -\frac{dV}{dx}$$

If the system in equilibrium then net force on the system is zero. So

$$F_x = 0$$

$$i.e. \quad \frac{dV_x}{dx} = 0$$

This is the condition of equilibrium. It gives the equilibrium points or positions. Let the variation of potential with distance x is shown in figure-1. In $V(x)$ versus $'x'$ graph the points where $\frac{dV}{dx} = 0$ i.e. at the point where tangent to the curve is parallel to x-axis are the equilibrium points. Therefore in figure, points A, B, C and D are equilibrium points.

## Stable Equilibrium Points:

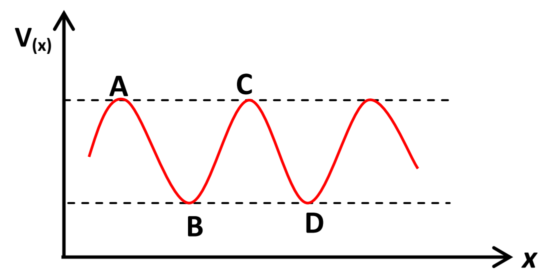
A point is stable equilibrium point if a particle at this point, when displaced towards right side, it experiences a force towards left and vice-versa. Therefore at stable equilibrium points

$$\frac{dF}{dx} < 0 \text{ i.e. } \frac{dF}{dx} = Negative$$

$$\because \quad F = -\frac{dV}{dx}$$

$$\therefore \quad \frac{d^2V}{dx^2} = Negative \times Negative = Positive$$

$$i.e. \quad \frac{d^2V}{dx^2} > 0$$




This is the condition of minima. Here point is the stable equilibrium points. Thus a stable equilibrium point is a minimum of $V(x)$ versus $'x'$ plot. Therefore points B and D are stable equilibrium points.

## Unstable Equilibrium Points:

A point is unstable equilibrium point if a particle at this point, when displaced towards right side, it experiences a force also in rightward direction. That is the force tries to displace the particle away from the equilibrium point. Therefore at unstable equilibrium points

$$\frac{dF}{dx} > 0 \text{ i.e. } \frac{dF}{dx} = Positive$$

$$\because \quad F = -\frac{dV}{dx}$$

$$\therefore \quad \frac{d^2V}{dx^2} = Positive \times Negative = Negative$$

$$i.e. \quad \frac{d^2V}{dx^2} < 0$$

This is the condition of maxima. Here point is the unstable equilibrium points. Thus an unstable equilibrium points in $V(x)$ versus $'x'$ plot graph are maxima points. Therefore points A and C are unstable equilibrium points.

Q: If $V(x) = \alpha x^4, \alpha > 0$ then find equilibrium point and check it is stable or unstable.

Solution: Let $x_0$ is equilibrium point. For equilibrium point

$$\left.\frac{dV(x)}{dx}\right|_{x=x_0} = 0$$

$$\Rightarrow \alpha \left.\frac{dx^4}{dx}\right|_{x=x_0} = 0$$

$$\Rightarrow 4\alpha\, x_0^3 = 0$$

$$\Rightarrow x_0 = 0$$

So $x_0 = 0$ is equilibrium point.

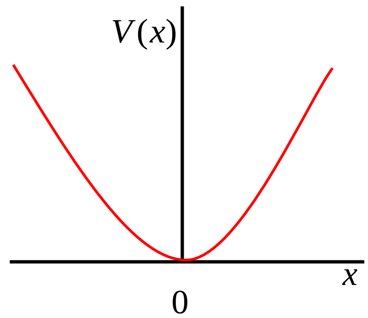
Now

$$\left.\frac{d^2V(x)}{dx^2}\right|_{x=x_0} = 12\alpha\, x_0^2\Big|_{x_0=0} \Rightarrow \frac{d^2V(x)}{dx^2} = 0$$

Further higher even derivative

$$\left.\frac{d^4V(x)}{dx^2}\right|_{x=x_0=0} = 24 > 0$$

So $x_0 = 0$ is stable equilibrium point.




$x_0 = 0$

(*Stable equilibrium point*)

Q: If $V(x) = \alpha x^3, \alpha > 0$ then find equilibrium point and check it is stable or unstable.

## Theory of small oscillations:

Let us consider a conservative system having $n$ degrees of freedom, described by a set of $n$ generalised co-ordinates $q_1, q_2, q_3, \cdots\cdots\cdots q_n$. The system has a stable equilibrium corresponding to the minimum of potential energy $V_0$. Let us assume that the generalised co-ordinates are measured with respect to this stable equilibrium position. Expanding the potential $V(q_1, q_2, q_3, \cdots\cdots\cdots q_n)$ of the system about the equilibrium point in a Taylor series, we have

$$V(q_1, q_2, q_3, \cdots\cdots\cdots q_n) = V_0 + \sum_i \left(\frac{\partial V}{\partial q_i}\right)_0 q_i + \frac{1}{2}\sum_i\sum_j \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 q_i q_j + \cdots\cdots\cdots\cdots \quad \ldots\ldots (1)$$

The first term is the potential energy at the equilibrium position which is a constant and may be taken as zero. The second term vanishes, since at the equilibrium position, the generalised force $(\partial V / \partial q_j)_0 = 0$. Neglecting higher terms, we have

$$V = \frac{1}{2}\sum_i\sum_j \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 q_i q_j$$

$$\Rightarrow V = \frac{1}{2}\sum_i\sum_j V_{ij}\, q_i q_j \qquad \text{----- (2)}$$

Where, $$V_{ij} = \left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0$$

If the transformation equations defining the generalised co-ordinates do not depend explicitly on time, the kinetic energy is a quadratic function of the generalised velocities. That is

$$T = \frac{1}{2}\sum_i\sum_j m_{ij}\, \dot{q}_i\, \dot{q}_j \qquad \text{----- (3)}$$

Where the $m_{ij}$'s are in general functions of the generalised co-ordinates and contain the masses. Expanding $m_{ij}$ into a Taylor series about the equilibrium values of $q_i$'s and neglecting terms beyond the constant values of $m_{ij}$ at the equilibrium position

$$m_{ij} = (m_{ij})_0 \qquad \text{----- (4)}$$

Designating the constant values of $(m_{ij})_0$ by the constant $G_{ij}$'s, we have

$$T = \frac{1}{2}\sum_i \sum_j G_{ij}\, \dot{q}_i\, \dot{q}_j \qquad \text{------ (5)}$$

Now, the Lagrangian of the system is

$$L = T - V = \frac{1}{2}\sum_i \sum_j G_{ij}\, \dot{q}_i\, \dot{q}_j - \frac{1}{2}\sum_i \sum_j V_{ij}\, q_i q_j$$

$$\Rightarrow L = \frac{1}{2}\sum_i \sum_j \left( G_{ij}\, \dot{q}_i\, \dot{q}_j - V_{ij}\, q_i q_j \right) \qquad \text{----- (6)}$$

$$\therefore \quad \frac{\partial L}{\partial \dot{q}_i} = \frac{1}{2}\sum_j G_{ij}\, \dot{q}_j \quad \text{and} \quad \frac{\partial L}{\partial q_i} = -\frac{1}{2}\sum_j V_{ij}\, q_j$$

Therefore, Lagrangian equation

$$\frac{d}{dt}\left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0$$

$$\Rightarrow \frac{d}{dt}\left( \frac{1}{2}\sum_j G_{ij}\, \dot{q}_j \right) - \left( -\frac{1}{2}\sum_j V_{ij}\, q_j \right) = 0$$

$$\Rightarrow \frac{1}{2}\sum_j G_{ij}\, \ddot{q}_j + \frac{1}{2}\sum_j V_{ij}\, q_j = 0$$

$$\Rightarrow \sum_j \left( G_{ij}\, \ddot{q}_j + V_{ij}\, q_j \right) = 0\,, \qquad \text{----- (7)}$$

Equation (7) is a system of $n$ second order homogeneous differential equation with constant coefficient. Let the solution of this differential equation be

$$q_j = C\, a_j\, e^{-i\omega t} \qquad \text{----- (8)}$$

$$\therefore \quad \dot{q}_j = C\, a_j\, (-i\omega) e^{-i\omega t}$$

And

$$\ddot{q}_j = C\, a_j\, (-i\omega)(-i\omega) e^{-i\omega t} = -\omega^2 C\, a_j e^{-i\omega t} \qquad \text{------ (9)}$$

Now, substituting the values of equation (8) and (9) in equation (7), we have

$$\sum_j \left( G_{ij}\, \ddot{q}_j + V_{ij}\, q_j \right) = 0$$

$$\Rightarrow \sum_j \left[ G_{ij}\left( -\omega^2 C\, a_j e^{-i\omega t} \right) + V_{ij}\, C\, a_j e^{-i\omega t} \right] = 0$$

$$\Rightarrow \sum_j \left( V_{ij} - \omega^2 G_{ij} \right) C\, a_j e^{-i\omega t} = 0$$

$$\therefore \quad \sum_j \left( V_{ij} - \omega^2 G_{ij} \right) a_j = 0 \qquad \because \quad C\, e^{-i\omega t} \neq 0 \qquad \text{---- (10)}$$

Expanding the equation

$$\left( V_{11} - \omega^2 G_{11} \right) a_1 + \left( V_{12} - \omega^2 G_{12} \right) a_2 + \left( V_{13} - \omega^2 G_{13} \right) a_3 + \cdots\cdots\cdots + \left( V_{1n} - \omega^2 G_{1n} \right) a_n = 0$$

$$\left( V_{21} - \omega^2 G_{21} \right) a_1 + \left( V_{22} - \omega^2 G_{22} \right) a_2 + \left( V_{23} - \omega^2 G_{23} \right) a_3 + \cdots\cdots\cdots + \left( V_{2n} - \omega^2 G_{2n} \right) a_n = 0$$

$$\left( V_{31} - \omega^2 G_{31} \right) a_1 + \left( V_{32} - \omega^2 G_{32} \right) a_2 + \left( V_{33} - \omega^2 G_{33} \right) a_3 + \cdots\cdots\cdots + \left( V_{3n} - \omega^2 G_{3n} \right) a_n = 0 \qquad \text{---- (11)}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\left( V_{n1} - \omega^2 G_{n1} \right) a_1 + \left( V_{n2} - \omega^2 G_{n2} \right) a_2 + \left( V_{n3} - \omega^2 G_{n3} \right) a_3 + \cdots\cdots\cdots + \left( V_{nn} - \omega^2 G_{nn} \right) a_n = 0$$

This set of $n$ linear homogeneous equations for the $a$'s will have a solution only if the determinant of the coefficients is zero. i.e.

$$\begin{vmatrix} (V_{11}-\omega^2G_{11})(V_{12}-\omega^2G_{12})\cdots\cdots\cdots(V_{1n}-\omega^2G_{1n}) \\ (V_{21}-\omega^2G_{21})(V_{22}-\omega^2G_{22})\cdots\cdots\cdots(V_{2n}-\omega^2G_{2n}) \\ (V_{31}-\omega^2G_{31})(V_{32}-\omega^2G_{32})\cdots\cdots\cdots(V_{3n}-\omega^2G_{3n}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (V_{n1}-\omega^2G_{n1})(V_{n2}-\omega^2G_{n2})\cdots\cdots\cdots(V_{nn}-\omega^2G_{nn}) \end{vmatrix} = 0 \qquad \text{----- (12)}$$

This equation is of $n^{th}$ degree in $\omega^2$ and the roots give the frequencies for which equation (8) represents a correct solution of equation (7). That is, the equations of motion will satisfied by an oscillatory solution of the type given in equation (8), not merely for one frequency but in general for a set of $n$ frequencies $\omega_p$. These frequencies are often called the frequencies of free vibration or the resonant frequencies of the system.

Therefore, complete solution of the equation of motion with all allowed frequencies is

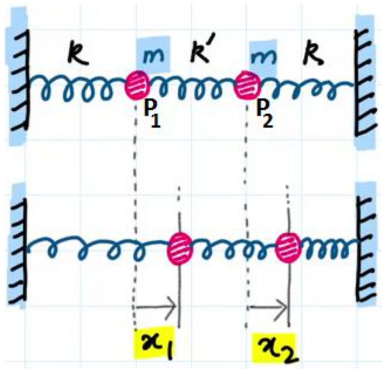
$$q_j = \sum_p C_p \, a_{jp} \, e^{-i\omega_p t} \qquad \text{------ (13)}$$

Each of the co-ordinates is dependent on all the frequencies and to determine the amplitudes $(a_j)$'s, each value of $\omega_p$ is substituted separately in equation (11).

## Normal Modes:

## Normal modes of coupled oscillator:

Let us consider a couple oscillators consist by two identical particles of masses "m" attached by three springs between two rigid supports. Let the spring constant of the two outer spring be $k$ and the inner spring be $k'$ . Let the particles $P_1$ and $P_2$ gets displace due action of external force through a distance $x_1$ and $x_2$ .The system has degree of freedom 2 (since system has two masses and one direction fixed) and the generalised coordinate's $q_1$ is $x_1$ and $q_2$ is $x_2$ .




Now, kinetic energy of the system

$$T = \frac{1}{2} m \, \dot{x}_1^2 + \frac{1}{2} m \, \dot{x}_2^2$$

And potential energy of the system

$$V = \frac{1}{2} k \, x_1^{\,2} + \frac{1}{2} k' \, (x_1 - x_2)^2 + \frac{1}{2} k \, (-x_2)^2$$

Now, Lagrangian of the system is

$$L = T - V = \frac{1}{2} m \, \dot{x}_1^2 + \frac{1}{2} m \, \dot{x}_2^2 - \frac{1}{2} k \, x_1^{\,2} - \frac{1}{2} k' \, (x_1 - x_2)^2 - \frac{1}{2} k \, x_2^{\,2}$$

Now,

$$\frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1 \text{ and } \frac{\partial L}{\partial x_1} = -kx_1 - k'(x_1 - x_2)$$

Therefore, Lagrangian equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0$$

$$\Rightarrow \frac{d}{dt}(m\dot{x}_1) - \{-kx_1 - k'(x_1 - x_2)\} = 0$$

$$\Rightarrow m\ddot{x}_1 + kx_1 + k'(x_1 - x_2) = 0 \qquad \text{--- (1)}$$

Again, w.r.to $x_2$ ,

$$\frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2 \text{ and } \frac{\partial L}{\partial x_2} = k'(x_1 - x_2) - kx_2$$

Therefore, Lagrangian equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

$$\Rightarrow \frac{d}{dt}(m\dot{x}_2) - \{k'(x_1 - x_2) - kx_2\} = 0$$

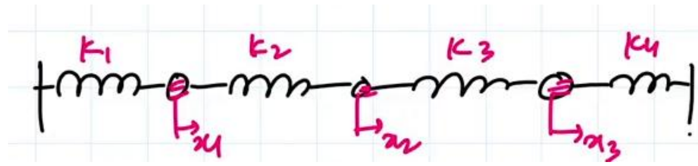
$$\Rightarrow m\ddot{x}_2 + kx_2 - k'(x_1 - x_2) = 0 \qquad \text{---- (2)}$$

In general, the potential energy for three particle system is

$$V = \frac{1}{2}k_1\, x_1^2 + \frac{1}{2}k_2\, (x_1 - x_2)^2 + \frac{1}{2}k_3\, (x_2 - x_3)^2 + \frac{1}{2}k_4\, \left(-x_3^2\right)$$

**OR**

$$V = \frac{1}{2}k_1\, x_1^2 + \frac{1}{2}k_2\, (x_2 - x_1)^2 + \frac{1}{2}k_3\, (x_3 - x_2)^2 + \frac{1}{2}k_4\, \left(-x_3^2\right)$$




From equation (1) and (2), we see that the solutions $x_1(t)$ and $x_2(t)$ of these equations is not necessarily SHM i.e. they are not executing SHM separately (individual masses are not executing SHM). To obtain the solution for the coupled oscillator, the above two ODE's need to be solved simultaneously.

Adding equation (1) and (2), we get

$$\Rightarrow m\ddot{x}_1 + kx_1 + k'(x_1 - x_2) + m\ddot{x}_2 + kx_2 - k'(x_1 - x_2) = 0$$

$$\Rightarrow m(\ddot{x}_1 + \ddot{x}_2) + k(x_1 + x_2) = 0$$

$$\Rightarrow (\ddot{x}_1 + \ddot{x}_2) + \frac{k}{m}(x_1 + x_2) = 0 \qquad \text{----- (3)}$$

Subtracting (2) from (1), we get

$$\Rightarrow m\ddot{x}_1 + kx_1 + k'(x_1 - x_2) - m\ddot{x}_2 - kx_2 + k'(x_1 - x_2) = 0$$

$$\Rightarrow m(\ddot{x}_1 - \ddot{x}_2) + k(x_1 - x_2) + 2k'(x_1 - x_2) = 0$$

$$\Rightarrow m(\ddot{x}_1 - \ddot{x}_2) + (k + 2k')(x_1 - x_2) = 0$$

$$\Rightarrow (\ddot{x}_1 - \ddot{x}_2) + \frac{k + 2k'}{m}(x_1 - x_2) = 0 \qquad \text{----- (4)}$$

Equation (3) corresponding to SHM of normal co-ordinates $(x_1 + x_2)$ with normal angular frequency

$$\omega_s = \sqrt{\frac{k}{m}}$$

Similarly, equation (4) is also corresponding SHM of normal co-ordinates $(x_1 - x_2)$ with normal angular frequency

$$\omega_f = \sqrt{\frac{k + 2k'}{m}}$$

Thus $x_1(t)$ and $x_2(t)$ are not necessarily executing any SHM but the linear combination of co-ordinates of the masses $x_1(t) + x_2(t)$ and $x_1(t) - x_2(t)$ are executing SHM.

Now, solution of equation (3) is

$$x_1(t) + x_2(t) = A_s Cos(\omega_s t + \phi_s) \qquad \text{---- (5)}$$

Where, $A_s$ and $\phi_s$ are the constants for amplitude and phase, which depends on initial conditions.

Similarly, solution of equation (4) is

$$x_1(t) - x_2(t) = A_f Cos(\omega_f t + \phi_f) \qquad \text{---- (6)}$$

Where, $A_f$ and $\phi_f$ are the constants for amplitude and phase, which depends on initial conditions.

Adding (5) and (6), we get

$$x_1(t)+x_2(t)+x_1(t)-x_2(t)=A_s Cos(\omega_s t+\phi_s)+A_f Cos(\omega_f t+\phi_f)$$

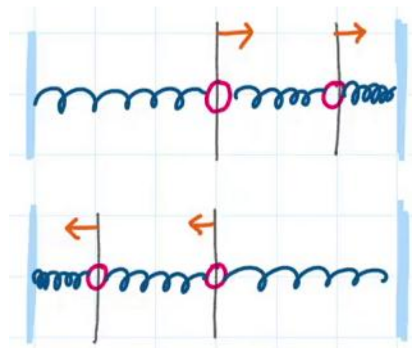
$$\Rightarrow x_1(t)=\frac{A_s}{2}Cos(\omega_s t+\phi_s)+\frac{A_f}{2}Cos(\omega_f t+\phi_f) \quad \text{--- (7)}$$

Subtracting (6) from (5), we get

$$x_1(t)+x_2(t)-x_1(t)+x_2(t)=A_s Cos(\omega_s t+\phi_s)-A_f Cos(\omega_f t+\phi_f)$$

$$\Rightarrow x_2(t)=\frac{A_s}{2}Cos(\omega_s t+\phi_s)-\frac{A_f}{2}Cos(\omega_f t+\phi_f) \quad \text{---- (8)}$$

Equations (7) and (8) are the general solutions or nature motions of 1st mass and 2nd mass respectively which are the linear combination of sine waves of frequencies $\omega_s$ and $\omega_f$ .



Let choose some initial conditions such that $A_f=0$ and $A_s=0$ .

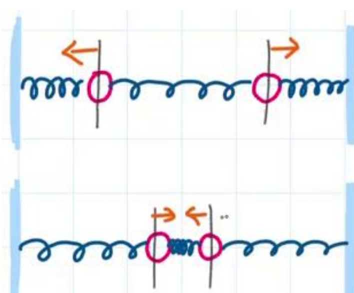
When $A_f=0$

$$x_1(t)=\frac{A_s}{2}Cos(\omega_s t+\phi_s)$$

And $$x_2(t)=\frac{A_s}{2}Cos(\omega_s t+\phi_s)$$

$$\therefore \quad x_1(t)=x_2(t)$$

i.e. Both $x_1(t)$ and $x_2(t)$ are executing SHM with same frequency $\omega_s$ and same amplitude in phase as shown in figure. This is called 1st normal mode.



When $A_s=0$

$$x_1(t)=\frac{A_f}{2}Cos(\omega_f t+\phi_f)$$

And $$x_2(t)=-\frac{A_f}{2}Cos(\omega_f t+\phi_f)$$

$$\therefore \quad x_1(t)=-x_2(t)$$

i.e. Both $x_1(t)$ and $x_2(t)$ are executing SHM with same frequency $\omega_f$ and opposite amplitude, $180^0$ out of phase as shown in figure. This is called 2nd normal mode.

## 9.1. INTRODUCTION

In this chapter, we generalize the harmonic oscillator problem of one degree of freedom in the Lagrangian formulation to the case of the small amplitude oscillations of a system of several degrees of freedom near the position of equilibrium. The theory of such small oscillations is extremely important in several areas of physics, *e.g.*, molecular spectra, acoustics, vibrations of atoms in solids, coupled mechanical oscillators and electrical circuits etc. When we go from a single oscillator to the problem of two coupled oscillators, the analysis results in some interesting and surprising new features. We shall see that the motion of two coupled oscillators in general is much complicated and none of the oscillators in general executes simple harmonic motion. However, for small amplitude oscillations, we may express the general motion as a superposition of two independent simple harmonic motions, both going on simultaneously. We call these two simple harmonic motions as ***normal modes*** or simply ***modes***. Further we shall see that a system of $N$ coupled oscillators with $N$ degrees of freedom, has exactly $N$ independent modes of vibrations and the general motion can be expressed as the superposition of $N$ normal modes. Each mode has its own frequency and wavelength. We will establish a relation between the wavelength and frequency of a mode, known as the ***dispersion relation***. Now, considering exceedingly large number of particles and allowing the interparticle distance to approach zero, we obtain the system as continuous medium and its motion is dealt as waves.

## 9.2. POTENTIAL ENERGY AND EQUILIBRIUM – *ONE DIMENSIONAL OSCILLATOR*

In order to understand the general theory of oscillations, it is essential to know about the potential energy at the equilibrium configuration. Let us consider a conservative system in which the potential energy is a function of position only. Let the system be specified by $n$ generalized coordinates $q_1, q_2, \ldots, q_n$, not involving time explicitly. For such a system, the potential energy is given by

$$V = V(q_1, q_2, \ldots, q_n) \qquad \ldots(1)$$

and the generalized forces are given by

$$G_k = -\frac{\partial V}{\partial q_k} \quad \text{where } k = 1, 2, \ldots, n \qquad \ldots(2)$$

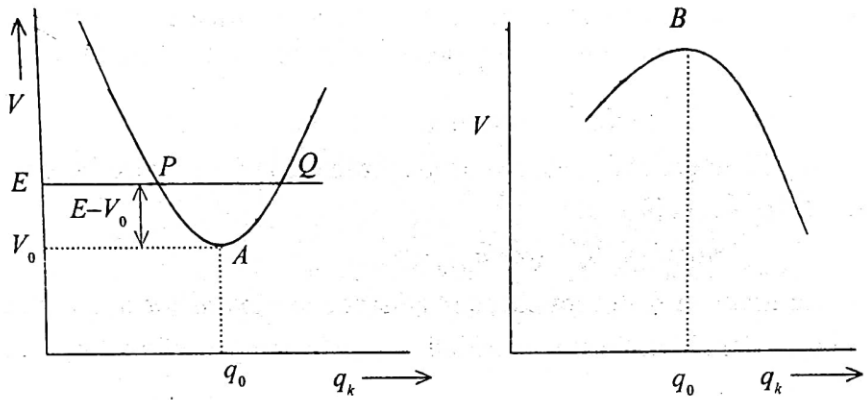
The system is said to be in equilibrium, if the generalized forces acting on the system are equal to zero, *i.e.*,

$$G_k = -\left[\frac{\partial V}{\partial q_k}\right]_0 = 0 \qquad \ldots(3)$$

Thus the potential energy has an extremum at the equilibrium configuration of the system, represented by the coordinates $q_1^0, q_2^0, \ldots, q_n^0$. Now, if the system is in equilibrium with zero initial velocities $\dot{q}_i$, the system will remain in equilibrium indefinitely. Examples of mechanical systems at equilibrium are a pendulum and a spring-mass system at rest, an egg standing on one end etc.

### 9.2.1. Stable, Unstable and Neutral Equilibrium

A system is said to be in ***stable equilibrium***, if a small displacement of the system from the rest position (by giving a little energy to it) results in a small bounded motion about the equilibrium position. In case, small displacement of the system from the equilibrium position results in an unbounded motion, it is in an ***unstable equilibrium***. Further, if the system on displacement has no tendency to move about or away the equilibrium position, it is said to be in ***neutral equilibrium***. An example of stable equilibrium is a pendulum in the rest position and that of an unstable equilibrium is an egg standing on one end. A coin placed flat anywhere on a table is in neutral equilibrium.




(a) Stable equilibrium (b) Unstable equilibrium

**Fig. 9.1 :** Potential energy curve

A graph drawn between the potential energy of the system and a particular coordinate $q_k$ is called ***potential energy curve*** and is shown in Fig. 9.1. The positions $A$ and $B$, where the generalized force $F = -\partial V/\partial q$ vanishes, are the positions of equilibrium; potential energy $V$ is minimum (say $V_0$) at $A$ [Fig 9.1($a$)] and maximum at $B$ [Fig. 9.1($b$)]. Position $A$ corresponds to the stable equilibrium, because if the system is displaced from $A$ to $Q$ by giving energy $(E - V_0)$ and left to itself, the system tries to come in the position of minimum potential energy. Consequently the potential energy will change to kinetic energy and at $A$ the energy $(E - V_0)$ will be purely in the kinetic form because of the conservation law. This will change again to potential form, when the system moves towards the position $P$ and hence a bounded motion ensues about the equilibrium position $A$. Obviously the position $B$ of the maximum potential energy represents the unstable equilibrium because any energy given to the system at this position will result more and more kinetic energy when the system moves either left or right to it. In this case, the system moves away from the equilibrium position. In case of neutral equilibrium, the potential energy is independent of the coordinate and equilibrium occurs at any arbitrary value of that coordinate.