Classical Mechanics of Point Particles

Review of Newtonian Mechanics:

Conservation of linear Momentum:

The linear momentum of a particle of mass 'm' and velocity 'v' is define as

$$p = mv$$

Therefore the net linear momentum for a system of n-particles is

$$P = \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} m_i v_i$$

From Newton's 2nd and 3rd law we have,

$$F_{ext} = \frac{dp}{dt}$$

i.e. the rate of change of linear momentum of a system of particle is equal to the net external force acting on the system.

If F_{ext} =0 then $\frac{dp}{dt} = 0$ and integrating we gate, P = constant.

Thus if the net external force acting on a system of particles is zero, the net linear momentum of the system remains constant. This is the principle of conservation of linear momentum.

Conservation of Angular Momentum:

The moment of linear momentum of a rotating particle is called the angular momentum. The angular momentum of a particle of linear momentum (p=mv) and having position vector r relative to and arbitrary origin is defined as

$$J = r \times p \qquad \text{------------------------- (1)}$$

For a system of n-particles, we have

$$J = \sum_i J_i = \sum_i r_i \times p_i \qquad \text{------------------------- (2)}$$

Differentiating above equation we get,

$$\frac{dJ}{dt} = \sum_i r_i \times \frac{dp_i}{dt} + \frac{dr_i}{dt} \times p_i$$

$$= \sum_i r_i \times \frac{dp_i}{dt} + 0$$

$$= \sum_i r_i \times \frac{dp_i}{dt} \qquad \text{------------------------ (3)}$$

Where $F_i = \frac{dp_i}{dt}$ = net force acting on i^{th} particle.

As internal force occurs is equal and opposite pair hence the net internal force acting on the system of particle is zero. Thus from equation (3) we have,

$$\frac{dJ}{dt} = \sum_i r_i \times F_i^{ext} = \tau_{ext} \qquad \text{----------------------- (4)}$$

Where $\tau_{ext} = \sum_i r_i \times F_i^{ext}$ is the torque arising due to external force only.

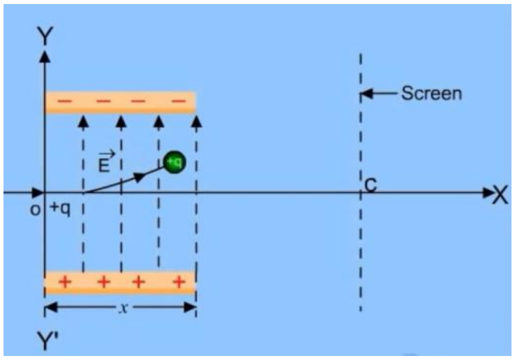
If $\tau_{ext} = 0$ then $\frac{dJ}{dt} = 0$

or J = constant.

Thus in the absence of an external torque the angular momentum of a system remains constant.

Motion of a charge particle in external electric field:

Let a charge particle $+q$ is thrown from 'o' towards the screen with a constant velocity 'u'. In the absence of electric field, charge particle directly strikes the screen at 'C'.



Now, motion along x-axis is

$$x = u_x t + \frac{1}{2} a_x t^2$$

$$\Rightarrow x = ut + 0 \qquad [\because a = 0]$$

$$\Rightarrow t = \frac{x}{u} \qquad \text{---- (1)}$$

Let a uniform electric field 'E' is applied. In the presence of electric field, the charge particle experience a force given by

$$F = qE \qquad \text{---- (2)}$$

According to Newton law

$$F = ma \qquad \text{--- (3)}$$

From equation (2) and (3), we get

$$ma = qE$$

$$a = \frac{qE}{m} \qquad \text{---- (4)}$$

It a constant acceleration of charge particle along y-axis since the electric field is constant.

Motion along y-axis is

$$y = u_y t + \frac{1}{2} a_y t^2$$

$$\Rightarrow y = 0 + \frac{1}{2} \frac{qE}{m} \frac{x^2}{u^2}$$

$$\Rightarrow y = \frac{1}{2} \frac{qE}{m} \frac{x^2}{u^2} \qquad \text{--- (5)}$$

Since, u, q, m, and E constant

$$\therefore y = kx^2$$

$$i.e. \quad y \propto x^2$$

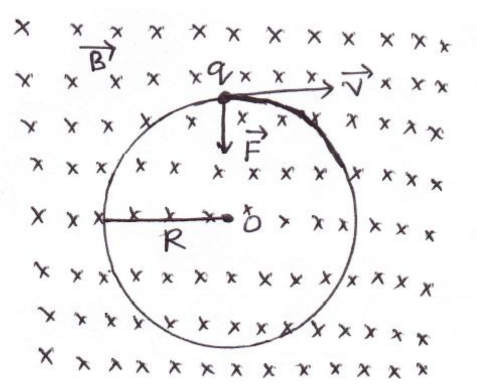
This is the equation of parabola; hence any charge particle thrown in uniform electric field is always describing parabolic path.

Motion of a charge particle in a uniform magnetic field:

(1). If a charge particle moves in the direction of a magnetic field then no force acts on it, because the velocity vector $\vec{v}$ and magnetic field vector $\vec{B}$ are parallel to each other.

(2). If a charge particle 'q' of mass 'm' moves with initial velocity $\vec{v}$ in a plane perpendicular to the direction of $\vec{B}$ then the charged particle experiences a force, which is given by

$$\vec{F} = q\left(\vec{v} \times \vec{B}\right) = qvB$$



Since the direction of this force always remains perpendicular to the velocity $\vec{v}$,

$$\therefore \quad \vec{F} \cdot \vec{v} = q\left(\vec{v} \times \vec{B}\right) \cdot \vec{v} = 0$$

$$\Rightarrow m \frac{d\vec{v}}{dt} \cdot \vec{v} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2} m v^2\right) = 0$$

$$\Rightarrow \frac{1}{2}mv^2 = \text{Constant.} \qquad \text{------- (1)}$$

This equation shows that the magnetic field does not work and does not changes the K.E. of the particle i.e. the force $\vec{F}$ cannot change the magnitude of velocity $\vec{v}$, but it changes only the direction of motion and hence the charge particles will moves in a circular path of radius 'R' in the magnetic field with constant speed as shown in figure. The force provides the necessary centripetal force.

$$\frac{mv^2}{R} = qvB$$

$$\Rightarrow R = \frac{mv}{qB} \qquad \text{---- (2)}$$

This is known as Gyro-radius. The radius with which a charge particle is travelling in circular orbit in magnetic field is called Gyro-radius.

$\therefore$ Angular velocity, $\omega = \frac{v}{R} = \frac{qB}{m}$ ------ (3) $[\because \; v = R\omega]$

This is known as Gyro-frequency.

$\therefore$ Frequency, $f = \frac{\omega}{2\pi} = \frac{qB}{2\pi\, m}$ ------ (4)

And Time period, $T = \frac{1}{f} = \frac{2\pi\, m}{qB}$ ------ (5)

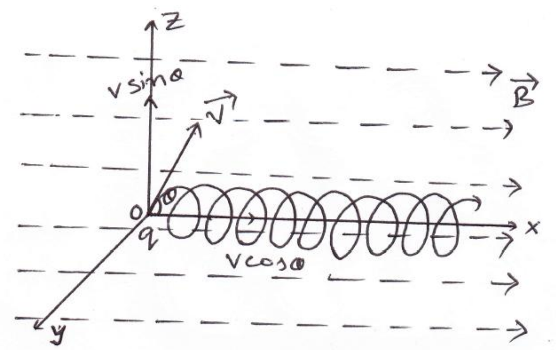
(3). If a charge particle 'q' of mass 'm' moves with initial velocity $\vec{v}$, making an angle 'θ' with the direction of uniform magnetic field $\vec{B}$ as shown in figure, then the velocity $\vec{v}$ can be resolved into two components:

(a). The horizontal component $\vec{v}_{\mathbf{B}}$ along the direction of the magnetic field is given by

$$\vec{v}_{\mathbf{B}} = v\cos\theta$$

(b). The perpendicular component $\vec{v}_N$ along the normal to the direction of magnetic field is given by

$$\vec{v}_N = v\sin\theta$$



Now, due to the normal component, the particle moves along a circular path of radius 'R' and the necessary centripetal force is provides by this component.

$$\therefore \quad \frac{mv_N^2}{R} = qv_N B$$

$$\Rightarrow R = \frac{mv_N}{qB} = \frac{mv\sin\theta}{qB} \qquad \text{----- (6)}$$

$\therefore$ Angular velocity, $\omega = \frac{v_N}{R} = \frac{v\sin\theta}{R}$ $[\because \; v = R\omega]$

$$\Rightarrow \omega = v\sin\theta \times \frac{qB}{mv\sin\theta} = \frac{qB}{m} \qquad \text{------ (7)}$$

$\therefore$ Frequency, $f = \dfrac{\omega}{2\pi} = \dfrac{qB}{2\pi\, m}$ ------ (8)

And Time period, $T = \dfrac{1}{f} = \dfrac{2\pi\, m}{qB}$ ------ (9)

For the component velocity $\vec{v}_{\mathbf{B}} = v\cos\theta$, there will be no force on the charge particle in the magnetic field, because the angle between $\vec{v}_{\mathbf{B}}$ and $\vec{B}$ is zero. Thus the particle covers the linear distance in the direction of the magnetic field with constant speed $v\cos\theta$.

Therefore, under the combined effect of the two component velocities, the charged particle in the magnetic field will cover linear path as well as circular path i.e. the path of the charge particle will be Helical, whose axis is parallel to the direction of magnetic field as shown in figure.

The linear distance covered by the charged particle in the magnetic field in time equal to one revolution of its circular path is known as pitch of helix and it is

$$d = v_B T = v\cos\theta \times \frac{2\pi\, m}{qB}$$

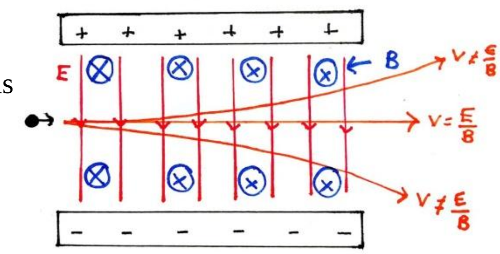
Motion of a charge particle in combined electric and magnetic field: (Velocity Selector)

Let us consider that a charged particle q is moving with a velocity v in combine electric and magnetic fields of intensities $\vec{E}$ and $\vec{B}$ respectively as shown in figure. Let the electric field $\vec{E}$ is along downward direction and the magnetic field $\vec{B}$ is perpendicular to the plan of the paper, directed inwards.

Now due to the electric field, the force on moving charge particle is

$$\vec{F}_E = q\vec{E}$$

Again due to the magnetic field, the force on moving charge particle is



$$\vec{F}_B = q\left(\vec{v} \times \vec{B}\right)$$

$$\Rightarrow \vec{F}_B = qvB\sin 90^0 = qvB$$

If the magnitude of electric field is greater than magnetic field then the charge particle tends to moves in downwards directions. Again, if the magnitude of magnetic field is greater than electric field then the charge particle tends to moves in the upwards directions. If the magnitude of electric and magnetic fields are equal then the charge particle is undeflected.i.e. Straight line.

Therefore, when

$$F_E = F_B$$

$$\Rightarrow qE = qvB$$

$$\Rightarrow v = \frac{E}{B}$$

This system of combined fields is known as a velocity selector or a velocity filter, because it is used for selecting a beam of electrons or ions having constant velocity $v = \dfrac{E}{B}$ and filtering out others by suitably adjusting the value of either field.

Motion of a charge particle in crossed electric and magnetic field:

Let us consider that a charged particle q of mass m, emitted at the origin with zero initial velocity into a region of uniform electric and magnetic fields. The electric field $\vec{E}$ is acting along z-axis and magnetic field $\vec{B}$ is along x-axis. The force $q\vec{E}$ due to electric field will act along z-axis and hence it moves with a velocity $\vec{v}$ along z-axis. As the charged particle is placed in a magnetic field $\vec{B}$, it will experience a force, $q(\vec{v}\times\vec{B})$ acting in the direction perpendicular to both $\vec{v}$ and $\vec{B}$. Hence the particle will bent. The resultant force known as Lorents force is thus given by

$$\vec{F}=q\left[\vec{E}+\left(\vec{v}\times\vec{B}\right)\right]$$

Since, $\vec{v}\times\vec{B}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ 0 & \dot{y} & \dot{z}\\ B & 0 & 0\end{vmatrix}=\hat{x}(0-0)+\hat{y}(B\,\dot{z}-0)+\hat{z}(0-B\,\dot{y})=B\,\dot{z}\,\hat{y}-B\,\dot{y}\,\hat{z}$

Therefore, $\vec{F}=q\left[\vec{E}+\left(\vec{v}\times\vec{B}\right)\right]=q\left[E\,\hat{z}+B\,\dot{z}\,\hat{y}-B\,\dot{y}\,\hat{z}\right]$ ----- (1)

According to Newtons 2nd law,

$$\vec{F}=m\,\vec{a}=m(\ddot{y}\,\hat{y}+\ddot{z}\,\hat{z}) \quad \text{---- (2)}$$

Now, equating equation (1) and (2), we have

$$q\left[E\,\hat{z}+B\,\dot{z}\,\hat{y}-B\,\dot{y}\,\hat{z}\right]=m(\ddot{y}\,\hat{y}+\ddot{z}\,\hat{z})$$

Now, equating the coefficient of $\hat{y}$ and $\hat{z}$, we have

$$qB\,\dot{z}=m\,\ddot{y}\Rightarrow\ddot{y}=\frac{qB}{m}\dot{z}\Rightarrow\ddot{y}=\omega\,\dot{z}\Rightarrow\ddot{y}=\omega\,\dot{z} \quad \text{---- (3)}$$

Where, $\omega=\frac{qB}{m}$ is called cyclotron frequency.

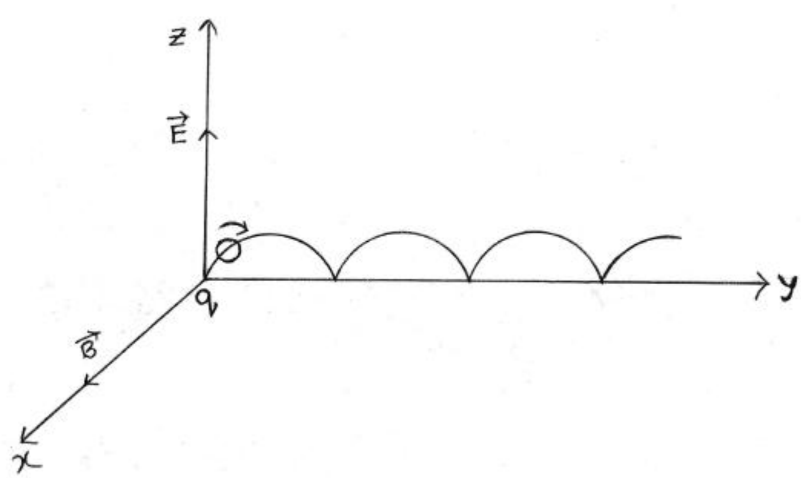
and $q(E-B\,\dot{y})=m\ddot{z}$

$$\Rightarrow\ddot{z}=\frac{qB}{m}\left(\frac{E}{B}-\dot{y}\right)$$

$$\Rightarrow qB\left(\frac{E}{B}-\dot{y}\right)=m\ddot{z}$$

$$\Rightarrow\ddot{z}=\frac{qB}{m}\left(\frac{E}{B}-\dot{y}\right)$$

$$\Rightarrow\ddot{z}=\omega\left(\frac{E}{B}-\dot{y}\right) \quad \text{---- (4)}$$



Differentiating equation (4), we get

$$\Rightarrow\dddot{z}=\omega(0-\ddot{y})$$

$$\Rightarrow\dddot{z}=-\omega\,\omega\,\dot{z} \qquad \because\ddot{y}=\omega\,\dot{z}$$

$$\Rightarrow\dddot{z}=-\omega^2\,\dot{z} \quad \text{---- (5)}$$

This is the differential equation of S.H.M. Its solution is

$$\dot{z}=v_z^{\max}\sin(\omega t+\phi) \quad \text{----- (6)}$$

At, $t=0$, $\dot{z}=0$ and $\phi=0$.

Therefore, $\dot{z}=v_z^{\max}\sin\omega t$ ------ (7)

Now to find v_z^{max} , differentiating equation (7), we get

$$\ddot{z} = v_z^{max}\omega\cos\omega t \quad \text{----- (8)}$$

From equation (4) and (8), we have

$$v_z^{max}\omega\cos\omega t = \omega\left(\frac{E}{B} - \dot{y}\right)$$

$$\Rightarrow v_z^{max}\cos\omega t = \left(\frac{E}{B} - \dot{y}\right) \quad \text{------- (9)}$$

At, $t = 0$, $\dot{y} = 0$ and $v_z^{max} = \frac{E}{B}$

Now, substituting the values of v_z^{max} in equation (9), we have

$$\frac{E}{B}\cos\omega t = \left(\frac{E}{B} - \dot{y}\right)$$

$$\dot{y} = \frac{E}{B}(1 - \cos\omega t) \quad \text{---- (10)}$$

This is the equation for $\dot{y}$. Now equation for $\dot{z}$, substituting the values of v_z^{max} in equation (7), we have

$$\dot{z} = \frac{E}{B}\sin\omega t \quad \text{---- (11)}$$

To get trajectory of the motion of the charge particle, integrating equation (10) and (11), we get

$$y = \frac{E}{B}\int_0^t (1 - \cos\omega t)dt$$

$$\Rightarrow y = \frac{E}{B}\left(t - \frac{\sin\omega t}{\omega}\right)$$

$$\therefore \quad y = \frac{E}{B\omega}(\omega t - \sin\omega t) \quad \text{----- (12)}$$

And $$z = \int_0^t \frac{E}{B}\sin\omega t\, dt$$

$$\Rightarrow z = \frac{E_0}{B_0}\left[-\frac{\cos\omega t}{\omega}\right]_0^t$$

$$\therefore \quad z = \frac{E}{B\omega}[1 - \cos\omega t] \quad \text{--- (13)}$$

Let, $R = \frac{E}{B\omega}$, from equations (12) and (13), we have

$$y = R(\omega t - \sin\omega t) \quad \text{and } z = R[1 - \cos\omega t]$$

$$\Rightarrow \frac{y}{R} - \omega t = -\sin\omega t \quad \text{and } \frac{z}{R} - 1 = -\cos\omega t$$

$$\Rightarrow \frac{y - R\omega t}{R} = -\sin\omega t \quad \text{and } \frac{z - R}{R} = -\cos\omega t$$

Squaring and adding these equations we have

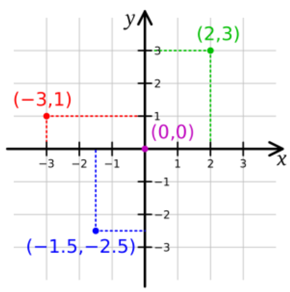
$$\frac{(y - R\omega t)^2}{R^2} + \frac{(Z - R)^2}{R^2} = 1$$

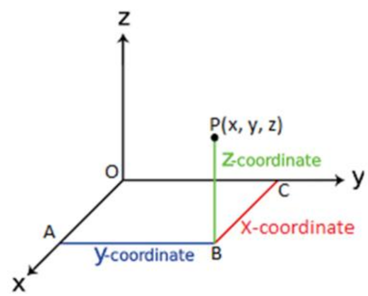
$$\Rightarrow (y - R\omega t)^2 + (Z - R)^2 = R^2 \quad \text{--- (14)}$$

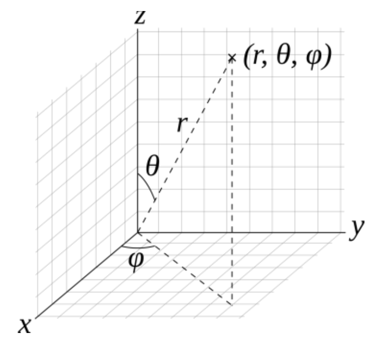
This is the trajectory for a cycloid motion, which is defined as the path generated by the point on the circumference of a circle, rolling along a circle as shown in figure. In the present case the radius of the rolling circle is $R=\frac{E}{B\omega}$, the maximum displacement along z-direction is $\frac{2E}{B\omega}$. The z-displacement becomes zero at $t=0,\frac{2\pi}{\omega},\frac{4\pi}{\omega}$. The y-displacement in time $t=\frac{2\pi}{\omega}$ is $\frac{2\pi E}{B\omega}$.

Coordinates:

A set of values that show an exact position of a particle or a system is called coordinates. Examples of coordinate are shown in figure below.







Constraints:

The limitation or geometrical restrictions on the motion of a particle or system of particles are generally known as constraints.

Example:

(1) The motion of point mass of a simple pendulum is restricted, since the point mass always remains at a constant distance from the point of suspension.

(2) The motion of a rigid body is always such that the distance between any two particles remains unchanged.

Classification of constraints:

The constraints may be classified as:

1. (a) Scleronomic: When the constraint are independent of time (i.e. if the constraint relation does not explicitly depend on time) then they are called scleronomic constraint.

Example: In case of rigid body.

(b) Rheonomic: When the constraint relations are explicitly depends on time then they are called Rheonomic constraints.

Example: When a particle (Bead) is made to slide on a moving wire.

2(a). Holonomic: A holonomic constraint is one that may be expressed in the form of an equation relating the co-ordinate of the system and time. (i.e. Constraint relation are or can be made independent of velocity.)

The general form of such equations for a system of N-particles is

$$F(r_1, r_2, r_3, \ldots\ldots r_N, t) = 0$$

Where $r_1, r_2, r_3, \ldots\ldots r_N$, be the the position co-ordinates of a system and "t" denotes the time.

Example:

(1) The constraints involved in the motion of rigid bodies in which the distance between any two particular points is always fixed are holonomic.

(2) The constraints involved when a particle is restricted to move along a curved or surface are holonomic.

(3) The constraint involved in the motion of the point mass of a simple pendulum are holonomic.

(b) Non-Holonomic: If the constraints cannot be expressed in the form of $F(r_1, r_2, r_3, \ldots\ldots r_N, t) = 0$, then they are called non-holonomic constraints.

Example: (i) The constraints involved in the motion of the particle placed on the surface of a solid sphere are non-holonomic. The condition of constraints in this case are expressed as

$$r^2 - a^2 \geq 0$$

Where "a" is the radius of sphere. This is an inequality and hence not in form of F(r_1, r_2, r_3, …………………………………r_N, t) =0.

(ii) The constraints involved in the motion of the molecules in a gas container are non-holonomic.

(iii) An object rolling on a rough surface without slipping involves non-holonomic constraint in the description of its motion.

Degree of freedom:

The minimum number of independent co-ordinates (variables) required to specify the position and all the possible configuration of a dynamical system which are compatible with the given constraints is called the number of degree of freedom.

Example:

(i) When a single particle moves in space, it has three degree of freedom but if it is constraint to move along a certain space curve, it has only one.

(ii) When a system of two particles, moving freely in space, requires two sets of three coordinates $\left[e.g. - (x_1, y_1, z_1)\ and\ (x_2, y_2, z_2)\right]$ i.e. six coordinates to specify its position. Thus the system has six degrees of freedom.

(iii) For system of N-particles moving independently of each other, the number of degree of freedom is 3N.

(iv) An N particle with $'n'$ constraint relations has $3N - n$ independent variables. Thus the number of degree of freedom is $f = 3N - n$.

Generalised Co-ordinates:

The minimum number of independent coordinates or variables which is required to describe the motion of a dynamical system is known as Generalised Coordinates.

A system of N particles free from constraints has 3N independent co-ordinates which can easily be described by Cartesian co-ordinates but in case of holonomic constraint $'n'$ the number of degree of freedom will be $f = 3N - n$ which is less than the total number of Cartesian co-ordinates involved. Hence in the presence of constraint, Cartesian co-ordinates fail to completely describe the configuration of the system. Therefore a different co-ordinate system was introduced which can be particularizes according the constraint relations known as Generalised co-ordinates.

The Cartesian coordinates can be expressed in terms of Generalised coordinates. If x_1, x_2, x_3, ………………..x_n be the Cartesian co-ordinates of a system of particle, then these Cartesian co-ordinates can be expressed as functions of Generalised co-ordinate q_1, q_2, q_3,……………………..q_n. i.e.,

$$x_1 = x_i(q_1, q_2, q_3, q_4, \ldots\ldots q_n, t)$$
$$x_2 = x_2(q_1, q_2, q_3, q_4, \ldots\ldots q_n, t)$$
$$\vdots$$
$$x_n = x_n(q_1, q_2, q_3, q_4, \ldots\ldots q_n, t)$$

Where "t" denotes the time. These equations are called Transformation equation.

In general, if r_i, $(i = 1,2,3,\ldots\ldots)$ be the Cartesian coordinates and q_n be the Generalised coordinates then

$$r_i = r_i(q_1, q_2, q_3, q_4, \ldots\ldots q_n, t)$$

Generalised Displacement:

Let us consider an N-particle system for which a small displacement $\partial \vec{r}_i$ is defined by change in position coordinates $\vec{r}_i, (i = 1,2,3,\ldots\ldots N)$ with time (t) kept as constant. The position vector $\vec{r}_i$ of the ith particle in the form of generalized coordinates can be written as

$$\vec{r}_i = \vec{r}_i(q_1, q_2, q_3, q_4, ...q_f, t)\text{, where } f = 3N - n$$

Using Euler's theorems

$$\partial\vec{r}_i = \frac{\partial\vec{r}_i}{\partial q_1}\delta q_1 + \frac{\partial\vec{r}_i}{\partial q_2}\delta q_2 + \frac{\partial\vec{r}_i}{\partial q_3}\delta q_3 + \cdots\cdots\cdots + \frac{\partial\vec{r}_i}{\partial q_f}\delta q_f + \frac{\partial\vec{r}_i}{\partial t}\delta t$$

$$\Rightarrow \partial\vec{r}_i = \sum_{k=1}^{f}\frac{\partial\vec{r}_i}{\partial q_k}\delta q_k \qquad (\text{Since} \because \delta t = 0)$$

Where, ∂q_k represents generalised displacement.

> Euler's theorems: If $f(r)$ be a function depends on 3-variables x, y, z then $f(r)$ can be written as
>
> $f(r) = f(x, y, z)$
>
> Now complete derivative of $f(r)$
>
> $$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$
>
> This equation can also be written as (small change in $f(r)$)
>
> $$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z$$

Generalised Velocity:

Let us consider an N-particle system. The generalised velocity $\dot{q}_k$ is the time derivative of the generalised coordinates q_k. The position vector $\vec{r}_i, (i = 1,2,3,........N)$ of the particle in the form of generalised coordinates and time (t) can be written as

$$\vec{r}_i = \vec{r}_i(q_1, q_2, q_3, q_4, ...q_f, t)$$

Using Euler's theorems

$$d\vec{r}_i = \frac{\partial\vec{r}_i}{\partial q_1}dq_1 + \frac{\partial\vec{r}_i}{\partial q_2}dq_2 + \frac{\partial\vec{r}_i}{\partial q_3}dq_3 + \cdots\cdots\cdots + \frac{\partial\vec{r}_i}{\partial q_f}dq_f + \frac{\partial\vec{r}_i}{\partial t}dt$$

Dividing both side by dt, we have

$$\frac{d\vec{r}_i}{dt} = \frac{\partial\vec{r}_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial\vec{r}_i}{\partial q_2}\frac{dq_2}{dt} + \frac{\partial\vec{r}_i}{\partial q_3}\frac{dq_3}{dt} + \cdots\cdots\cdots + \frac{\partial\vec{r}_i}{\partial q_f}\frac{dq_f}{dt} + \frac{\partial\vec{r}_i}{\partial t}\frac{dt}{dt}$$

$$\Rightarrow \dot{\vec{r}}_i = \vec{V}_i = \sum_{k=1}^{f}\frac{\partial\vec{r}_i}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial\vec{r}_i}{\partial t}$$

$$\Rightarrow \vec{V}_i = \sum_{k=1}^{f}\frac{\partial\vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial\vec{r}_i}{\partial t}$$

Where, $\dot{q}_k$ is called generalised velocity.

(Generalised velocity associated with a particular Generalised co-ordinate q_k may be described in terms of its time derivative $\dot{q}_k$.)

Virtual displacement: A virtual displacement of a system refers to a change in the configuration of the system as the result of any infinitesimal change of the coordinates ∂r_i consistent with the forces and constraints imposed on the system at the given instant'‘t’. The displacement is called virtual to distinguish it from an actual displacement of the system occurring in a time interval ‘dt’, during which the constraints and forces may be changing. In other words, any imaginary displacement which is consistent with the constraint relations at a given instant (i.e. without allowing the real time to change) is called a virtual displacement. So by definition, a virtual infinitesimal displacement is given by

$$\partial r_i = dr_i\big|_{dt=0}$$

Virtual work: Suppose a particle is subjected to a force F. If the force produces a virtual displacement ∂r, then the work performed by the force is termed as virtual work. The following equation defines the virtual work

$$\partial W = F.\partial r$$

The force F is the sum of the constraint force ‘f’ (say) and the applied force F^a.

D'Alemberts Principle:

This principles states that a system of moving particles can be considered to be in equilibrium under the action of the external force plus an additional force $-\dot{P}$ (- rate of change of momentum) which is known as the reversed effective force or inertia force.

Let us consider any number of force be applied to the i^th particle of the system. If the system is in equilibrium, then the total force acting on it is zero. i.e.,

$$\sum_i F_i \partial r_i = 0 \qquad \text{--------------- (1)}$$

Where ∂r_i is the virtual displacement and F_i is the equilibrium force acting on each particle of the system . The force F_i can be written as the sum of the external force i.e., applied force plus constraint force i.e.,

$$F_i = F_i^a + f_i \qquad \text{--------------- (2)}$$

From equation (1), we have

$$\sum_i (F_i^a + f_i).\partial r_i = 0$$

$$\Rightarrow \sum_i F_i^a \partial r_i + \sum_i f_i \partial r_i = 0$$

If the constraint force is normal to the motion i.e., $f_i \perp \partial r_i$ then $\sum_i f_i \partial r_i = 0$

$$\therefore \sum_i F_i^a \partial r_i = 0 \qquad \text{--------------- (3)}$$

Thus for an equilibrium of a system, the virtual work of the applied forces vanishes. This is known as the principle of virtual work. This principle was developed by D'Alemberts by taking

$$F_i = \dot{P}_i \Rightarrow F_i - \dot{P}_i = 0$$

Which states that the particle in the system will be in equilibrium under a force equal to the actual force plus a reverse effective force $-\dot{P}$. Thus from equation (1), we have

$$\sum_i (F_i - \dot{P}_i).\partial r_i = 0$$

$$\Rightarrow \sum_i \left\{ (F_i^a + f_i) - \dot{P}_i \right\}.\partial r_i = 0$$

$$\Rightarrow \sum_i \left\{ (F_i^a - \dot{P}_i).\partial r_i \right\} + \sum_i f_i \partial r_i = 0$$

$$\Rightarrow \sum_i \left\{ (F_i^a - \dot{P}_i).\partial r_i \right\} + 0 = 0$$

$$\therefore \sum_i \left\{ (F_i^a - \dot{P}_i).\partial r_i \right\} = 0 \qquad \text{---------------- (4)}$$

Which is the D'Alemberts principle .

Lagrangian equation of motion from D'Alemberts Principles:

This is the equation of motion in the form of generalised coordinates. The transformation from old coordinate r_i to new generalised coordinate q_i starts from transformation equations

$$r_i = r_i(q_1, q_2, q_3, \cdots\cdots\cdots q_n, t)$$

The virtual displacement is

$$\therefore \partial r_i = \frac{\partial r_i}{\partial q_1}\partial q_1 + \frac{\partial r_i}{\partial q_2}\partial q_2 + \frac{\partial r_i}{\partial q_3}\partial q_3 + \cdots\cdots\cdots\cdots + \frac{\partial r_i}{\partial q_n}\partial q_n$$

$$\Rightarrow \partial r_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j}\partial q_j \qquad \text{------------------- (1)}$$

And $V_i = \dot{r}_i = \frac{\partial r_i}{\partial q_1}\dot{q}_1 + \frac{\partial r_i}{\partial q_2}\dot{q}_2 + \frac{\partial r_i}{\partial q_3}\dot{q}_3 + \cdots\cdots\cdots + \frac{\partial r_i}{\partial q_n}\dot{q}_n + \frac{\partial r_i}{\partial t}$

$$= \sum_j \frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t} \qquad \text{-------------------- (2)}$$

Using the Einstein's summation conventions, the first term of the equation (2) can be written as

$$\frac{\partial r_i}{\partial q_j} = \frac{\partial \dot{r}_i}{\partial \dot{q}_j} \qquad \text{-------------------- (3)}$$

Also, $\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) = \frac{\partial}{\partial q_j}\frac{dr_i}{dt} = \frac{\partial \dot{r}_i}{\partial q_j}$ -------------------- (4)

Now according to the D'Alemberts Principles,

$$\sum_i \left\{(F_i^a - \dot{P}_i).\partial r_i\right\} = 0$$

$$\Rightarrow \sum_i F_i^a \partial r_i - \sum_i \dot{P}_i \partial r_i = 0 \qquad \text{------------------------------ (5)}$$

The first term of equation (5) can be written as

$$\sum_i F_i^a \partial r_i = \sum_{ij} F_i^a \frac{\partial r_i}{\partial q_j}\partial q_j$$

$$= \sum_j Q_j \partial q_j \qquad \text{------------------------------ (6)}$$

Where $Q_j = \sum_{ij} F_i^a \frac{\partial r_i}{\partial q_j}$ is the generalized force.

Also the second term of equation (5) can be written as

$$\sum_i \dot{P}_i \partial r_i = \sum_i m_i \ddot{r}_i \partial r_i = \sum_{ij} m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j}\partial q_j \qquad \text{--------------------------------- (7)}$$

The co-efficient of ∂q_j can be expressed as

$$\sum_{ij} m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j}$$

$$= \sum_{ij} \frac{d}{dt}\left(m_i \dot{r}_i \frac{\partial r_i}{\partial q_j}\right) = \sum_{ij} m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} + \sum_{ij} m_i \dot{r}_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right)$$

$$\Rightarrow \sum_{ij} m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} = \sum_{ij} \frac{d}{dt}\left(m_i \dot{r}_i \frac{\partial r_i}{\partial q_j}\right) - \sum_{ij} m_i \dot{r}_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right)$$

$$= \sum_{ij} \frac{d}{dt}\left(m_i \dot{r}_i \frac{\partial \dot{r}_i}{\partial \dot{q}_j}\right) - \sum_{ij} m_i \dot{r}_i \left(\frac{\partial \dot{r}_i}{\partial q_j}\right)$$

$$= \sum_{ij} \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}m_i \dot{r}_i^2\right)\right\} - \sum_{ij} \frac{\partial}{\partial q_j}\left(\frac{1}{2}m_i \dot{r}_i^2\right)$$

$$\Rightarrow \sum_{ij} m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} = \sum_{ij} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \left(\frac{\partial T}{\partial q_j}\right)$$

From equation (7), we have

$$\sum_i \dot{P}_i \partial r_i = \left[\sum_j \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \left(\frac{\partial T}{\partial q_j}\right)\right]\partial q_j \quad \text{-------------------------- (8)}$$

Where $T = \frac{1}{2} m_i \dot{r}_i^2$ is the total energy of the system.

Substituting equation (6) & (8) in equation (5). We get

$$\sum_j Q_j \partial q_j - [\sum_j \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \left(\frac{\partial T}{\partial q_j}\right)]\partial q_j = 0$$

$$\Rightarrow \sum_j \left\{ Q_j - \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right]\right\}\partial q_j = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \quad \text{---------------------------- (9)}$$

Equation (9) is known as the Lagrangian equation of motion second kind for non-conservative system.

Case. (i) For a Conservative system: (If $Q_j = Q_j(q_1, q_2, \ldots\ldots q_n, t)$**)**

For a conservative system the force can be written as the gradient of some scalar function, i.e.,

$$F_i^a = -grad\ V$$

$$\Rightarrow F_i^a = -\nabla_i V \ ,$$

where V is a scalar function.

Since, the generalized force, Q_j is

$$Q_j = F_i^a \frac{\partial r_i}{\partial q_j} = -\nabla_i V \frac{\partial r_i}{\partial q_j} = -\frac{\partial V}{\partial r_i}\frac{\partial r_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

Hence Lagrangens equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T - V) = 0 \quad \text{------------------------ (10)}$$

Since V is a function of position only and independent of generalised velocity i.e. $\frac{\partial V}{\partial \dot{q}_j} = 0$

$$\therefore \frac{\partial T}{\partial \dot{q}_j} = \frac{\partial (T - V)}{\partial \dot{q}_j}$$

Now from equation (10), we have

$$\frac{d}{dt}\left(\frac{\partial (T - V)}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T - V) = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad \text{------------------ (11)}$$

Where $L = T - V$, $L(q, \dot{q}, t)$ is called Lagrangian of the system and the equation (11) is called Eular Lagrangian equation of motion.

Case: (ii):- If $\left\{Q_j = Q_j\left(q_1, q_2, \cdots\cdots q_n, \dot{q}_1, \dot{q}_2, \cdots\cdots \dot{q}_n, t\right)\right\}$

When the Q_j's are derivative from a potential energy U, then

$$Q_j = \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) - \frac{\partial U}{\partial q_j}$$

Where $U = U\left(q_1, \cdots\cdots q_n, \dot{q}_1 \cdots\cdots \dot{q}_n, t\right)$ is called generalized potential or velocity-dependent potential energy function, Then from Lagrangens equation for non-conservative system, we have from equation (9)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) - \frac{\partial U}{\partial q_j}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j}(T - U)\right) - \frac{\partial}{\partial q_j}(T - U) = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad \text{----------------------- (12)}$$

Where L = T- U is called the Lagrangian of the system.

Simple Pendulum:

Let "θ" be the angular displacement of the simple pendulum from the equilibrium position. If "l" be the effective length of the pendulum and "m" be the mass of the bob, then the displacement along arc OA = S is given by

$$S = l\theta \qquad \left[\because \theta = \frac{Arc}{Radius} = \frac{S}{l}\right]$$

Kinetic energy $T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2 \qquad \left[\because v = \frac{ds}{dt} = \frac{d(l\theta)}{dt} = \frac{l d\theta}{dt} = l\dot{\theta}\right]$

If the potential of the system, when the bob is at O, is zero, then the potential energy, when the bob is at A is given by

$$V = mg(OB) = mg(OC - BC) = mg(l - l\cos\theta) = mgl(1 - \cos\theta)$$

Hence $L = T - V$

$$\Rightarrow L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$$

Now $\frac{\partial L}{\partial \theta} = -mgl\sin\theta$ and $\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$

Substituting these value in the Lagrange's equation (here there is only one generalized coordinate $q_1 = \theta$)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

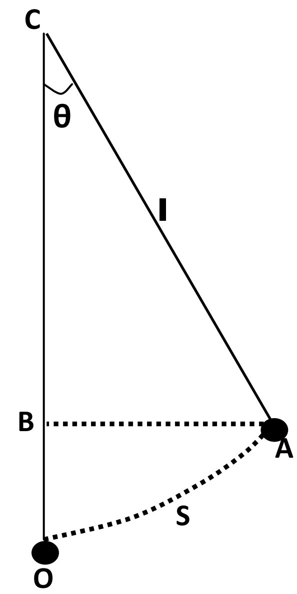
We get,

$$\frac{d}{dt}\left[ml^2\dot{\theta}\right]+mgl\sin\theta=0$$

$$\Rightarrow ml^2\ddot{\theta}+mgl\sin\theta=0$$

$$\Rightarrow l\ddot{\theta}+g\sin\theta=0$$

$$\Rightarrow \ddot{\theta}+\frac{g}{l}\sin\theta=0$$



This represents the equation of motion of a simple pendulum.
For small amplitude oscillation, $\sin\theta\cong\theta$,
Therefore the equation of motion of a simple pendulum is

$$\ddot{\theta}+\frac{g}{l}\theta=0$$

This represents a simple harmonic motion of period, given by

$$T=2\pi\sqrt{\frac{l}{g}}$$

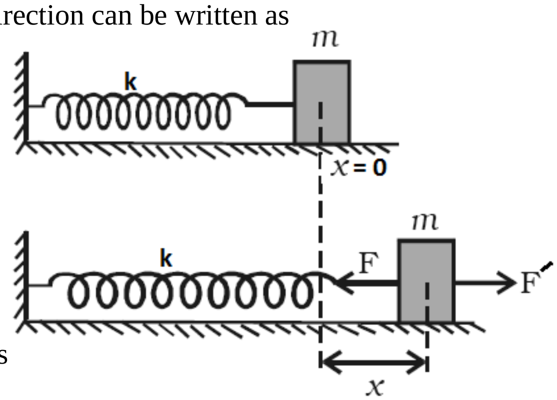
Linear Harmonic Oscillator:

A linear harmonic oscillator is a system of particles of mass vibrating under the action of force. In classical mechanics, a harmonic oscillator is a system that, when displaced from its equilibrium position, experiences a restoring force F proportional to the displacement $'x'$. Let us consider a block of mass 'm' attached with a spring. Let us displace the block from its mean position (equilibrium position $x=0$) by applying an external force F'. Due to spring force 'F', the block will execute linear harmonic motion. We assume that there is no any dissipation of energy during SHM and the spring obeys Hook's law. Hence

$$F=-k\,x \qquad \text{----------------------- (1)}$$

Where k is spring constant.
Now, Lagrange's equation of motion for one dimensional motion, say in "x" direction can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right)-\frac{\partial L}{\partial x}=0 \qquad \text{----------------------- (2)}$$



The kinetic energy of the system is

$$T=\frac{1}{2}mv^2$$

$$=\frac{1}{2}m\dot{x}^2$$

As we know that the spring force is a conservative force. So it can be written as

$$F=-\nabla V=-\frac{dV}{dx}$$

$$dV=-Fdx$$

Therefore, potential energy,

$$V=-\int F.dx \qquad \left[\because F=-\nabla V=-\frac{dV}{dx}\right]$$

$$\Rightarrow V=-\int -kx.dx \qquad \left[\because F=-kx\right]$$

$$\Rightarrow V=\frac{1}{2}kx^2+C$$

Where "C" is a constant of integration, which depends upon the initial conditions and "k" is spring constant. If we choose the horizontal plane passing through the position of equilibrium as the reference level, then $V=0$ at $x=0$ so that $C=0$. Thus Lagrangian is

$$L = T - V$$

$$\Rightarrow L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$\therefore \frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \text{and} \qquad \frac{\partial L}{\partial x} = -kx$$

Hence equation (2) takes the form

$$\frac{d}{dt}\left(m\dot{x}\right) - \left(-kx\right) = 0$$

$$\Rightarrow m\ddot{x} + kx = 0$$

This is the required expression. It is an equation of simple harmonic motion and can be put in the form

$$\ddot{x} + \frac{k}{m}x = 0$$

$$\Rightarrow \ddot{x} + w^2 x = 0$$

Where ω is the frequency of oscillation, given by

$$\omega = \sqrt{\frac{k}{m}}$$

Therefore,

Time period, $T = \frac{2\pi}{\omega}$

$$\Rightarrow T = 2\pi\sqrt{\frac{m}{k}}$$

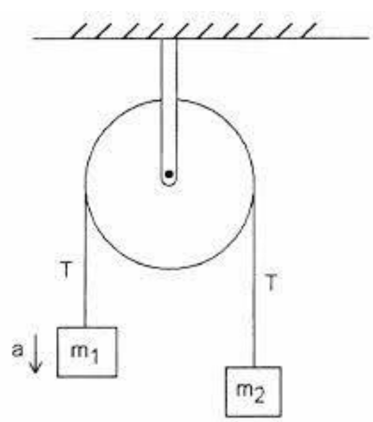
Atwood Machine:

Atwood machine was invented in 1784 by the English Mathematician 'George Atwood' to verify the mechanical laws of motion with constant acceleration.

The ideal Atwood machine consists of two objects of mass m_1 and m_2 , connected by an inextensible massless string over an ideal massless pulley. If $m_1 = m_2$ then it is in neutral equilibrium i.e. net force is zero. If $m_1 \neq m_2$ then the acceleration of the object is constant i.e. uniform acceleration.

Equation for constant acceleration:

Let us consider a system of two masses m_1 and m_2 , $(m_1 > m_2)$ suspended over a frictionless and massless pulley of radius 'r' and connected by a massless, inextensible, flexible string of constraints. So, the only forces we have to consider are: Tension force (T) and weight of the two masses $w_1 = m_1 g$ and $w_2 = m_2 g$.



Force affecting on m_1 ,

$$m_1 g - T = m_1 a \qquad \text{---- (1)}$$

Force affecting on m_2 ,

$$T - m_2 g = m_2 a \qquad \text{---- (2)}$$

Adding equation (1) and (2), we get

$$m_1 g - T + T - m_2 g = m_1 a + m_2 a$$

$$\Rightarrow (m_1 - m_2)g = (m_1 + m_2)a$$

$$\Rightarrow a = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)g \qquad \text{---- (3)}$$

Equation for Tension:

Using equation (1) and (3), we get

$$m_1 g - T = m_1 a$$

$$\Rightarrow T = m_1 g - m_1 a$$

$$\Rightarrow T = m_1 g - m_1 \times \left(\frac{m_1 - m_2}{m_1 + m_2} \right) g$$

$$\Rightarrow T = m_1 g \left(1 - \frac{m_1 - m_2}{m_1 + m_2} \right)$$

$$\Rightarrow T = m_1 g \left(\frac{m_1 + m_2 - m_1 + m_2}{m_1 + m_2} \right)$$

$$\Rightarrow T = \left(\frac{2m_1 m_2}{m_1 + m_2} \right) g$$

Lagrangian of Atwood machine:

Let us consider a system of two masses m_1 and m_2, $(m_1 > m_2)$ suspended over a frictionless and massless pulley connected by a massless, inextensible, flexible string of constraints of length l .

Now, velocities of two masses are

$$v_1 = \frac{dx}{dt} = \dot{x} \qquad \text{and} \qquad v_2 = \frac{d(l-x)}{dt} = -\dot{x}$$

Therefore, Kinetic energy

$$T = \frac{1}{2} m_1 {v_1}^2 + \frac{1}{2} m_2 {v_2}^2$$

$$\Rightarrow T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2$$

$$\Rightarrow T = \frac{1}{2} (m_1 + m_2) \dot{x}^2$$

Potential energy of the system with reference to the pulley

$$V = -m_1 g x - m_2 g (l - x)$$

Thus, the Lagrangian is given by

$$L = T - V$$

$$\Rightarrow L = \frac{1}{2} (m_1 + m_2) \dot{x}^2 + m_1 g x + m_2 g (l - x)$$

$$\therefore \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2) \dot{x} \qquad \text{and} \qquad \frac{\partial L}{\partial x} = (m_1 - m_2) g$$

Now, Lagrange's equation of motion for one dimensional motion is

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow \frac{d}{dt} (m_1 + m_2) \dot{x} - (m_1 - m_2) g = 0$$

$$\Rightarrow (m_1 + m_2) \ddot{x} - (m_1 - m_2) g = 0$$

$$\Rightarrow \ddot{x} = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) g$$

This is the equation of acceleration of Atwood machine.