Oscillation

Periodic motion:

A motion which repeats itself over and over again after a regular interval of time is called a periodic motion. The regular interval of time, after which the periodic motion is repeated again, is called its time period.

Example:

1. Rotation of earth around the sun.
2. Rotation of an electron around the nucleus.
3. Vibration of a simple pendulum.
4. Vibration of a loaded spring.
5. Vibration of a stretched spring.

Oscillatory motion:

A motion that repeats itself over and over again after a regular interval of time about its mean position within two well defined limits (called extreme position) on either side of the mean position is called oscillatory or vibratory motion.

All oscillatory motions are periodic motions but all the periodic motions are not oscillatory.

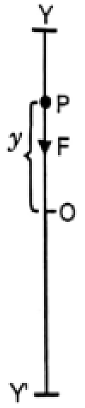
Example:

1. Vibration of a simple pendulum.
2. Vibration of a loaded spring.
3. Motion of a liquid column in a U-tube.
4. Motion of a body dropped in tunnel along the diameter of the earth.

Simple Harmonic Motion: (SHM)

A simple harmonic motion is an oscillatory motion in which the restoring force is proportional to the displacement from the mean position and is directed towards it.

Let 'y' be the displacement of a vibrating particle from the mean position at any instant 't'. In the case of a particle executing SHM, force is proportional to the displacement and is always oppositely directed to it.



That is $F \propto -y$

$$m\frac{d^2y}{dt^2} = -ky$$

Where k is the proportionality constant is referred to as the spring constant or the stiffness factor or force constant.

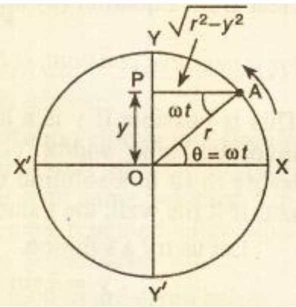
Or $$\frac{d^2y}{dt^2} = -\frac{k}{m}y$$

$$\Rightarrow \frac{d^2y}{dt^2} + \omega^2 y = 0,$$ Where $\omega^2 = \frac{k}{m}$.

This equation is differential equation of SHM.

Relation between SHM and uniform circular motion:

Let us consider a particle 'A' undergoing uniform circular motion in a circle having $XOX^/$ and $YOY^/$ as the horizontal and vertical diameters respectively. Let 'P' be the foot of perpendicular drawn from A upon one of the diameters, say vertical. 'P' is called Projection of A or Shadow of A. While A is at X, its projection P is at O. As A moves from X to Y, 'P' moves from O to Y along the vertical diameter. As A moves from Y to $X^/$, P comes back from Y to O. Thus, A completes its journey along the circumference of the circle, its projection moves from O to Y, Y to O, O to $Y^/$ and $Y^/$ to O. If particle A keep on moving, continuously, in uniform circular motion, its projection P keeps on vibrating to and fro about O. Motion of P along $YOY^/$ is called SHM.



Simple harmonic motion is defined as the projection of uniform circular motion on the diameter of circle of reference. Centre O of the circle of reference is called the mean position or neutral position.

Some definitions:

Displacement: Displacement of a particle vibrating in SHM, at any instant, is defined as its distance from the mean position at that instant.

Let P be the position of projection of A, at any instant of time 't'.

In ΔOAP, $\frac{OP}{AO} = \sin\theta \Rightarrow OP = OA\,Sin\theta$ Or $y = r\,Sin\theta$

Where 'r' is called the amplitude of vibrations. (Amplitude of a particle, vibrating in SHM is defined as its maximum displacement on either side of the mean position.)

Velocity: Home Work

Acceleration: Home Work

Differential equation of motion executing SHM:

(a) Let 'y' be the displacement from the mean position of rest at any instant 't'. In the case of a particle executing SHM, force is proportional to the displacement and is always oppositely directed to it.

That is $F \propto -y$

$$m\frac{d^2y}{dt^2} = -\mu y$$

Where μ is the proportionality constant is referred to as the spring constant or the stiffness factor.

Or $$\frac{d^2y}{dt^2} = -\frac{\mu}{m}y$$

$$\Rightarrow \frac{d^2y}{dt^2} + \omega^2 y = 0,\qquad \text{Where } \omega^2 = \frac{\mu}{m}.$$

(b) From energy consideration:

In any conservative system, the sum of the kinetic and potential energies is constant. The particle is displaced through a distance y in time t during SHM.

Kinetic energy $= \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{dy}{dt}\right)^2$

Potential energy is the total work the particle can do in changing the position. If dy be the small displacement then

Potential energy $= Fdy$

$= \mu y\,dy$

When it is displaced through y,

Potential energy $= \int_0^y \mu y dy = \frac{\mu y^2}{2}$

Total energy $= \frac{1}{2}m\left(\frac{dy}{dt}\right)^2 + \frac{\mu y^2}{2} =$ Constant

Differentiating both sides with respect to t, we get

$$\frac{1}{2}m.2\frac{dy}{dt}.\frac{d^2y}{dt^2} + \frac{\mu}{2}2y\frac{dy}{dt} = 0$$

Or $$\frac{dy}{dt}(m\frac{d^2y}{dt^2} + \mu y) = 0$$

Or $$(m\frac{d^2y}{dt^2} + \mu y) = 0$$

Or $$\frac{d^2y}{dt^2} + \frac{\mu}{m}y = 0$$

$$\therefore \frac{d^2y}{dt^2} + \omega^2 y = 0,\quad \text{Where } \omega^2 = \frac{\mu}{m}.$$

This is the required equation of motion.

5.3 SOLUTION OF DIFFERENTIAL EQUATION OF SIMPLE HARMONIC MOTION

Differential equation of simple harmonic motion is

$$\frac{d^2y}{dt^2} = -\frac{k}{m}y \quad \text{or} \quad \frac{d^2y}{dt^2} = -\omega^2 y$$

where $\omega = \sqrt{\frac{k}{m}}$ is a constant which depends upon the mass of the body in S.H.M. and the nature of the medium in which it is vibrating

Multiply both sides by $2\frac{dy}{dt}$,

$$2\frac{dy}{dt}\frac{d^2y}{dt^2} = -\omega^2 \,.\, 2y\frac{dy}{dt}$$

or
$$\frac{d}{dt}\left[\frac{dy}{dt}\right]^2 = \frac{d}{dt}(-\omega^2 y^2)$$

Integrating with respect to t,

$$\left(\frac{dy}{dt}\right)^2 = -\omega^2 y^2 + C$$

Where 'C' is the constant of integration.

In case of S.H.M., if $y = \pm A$ (amplitude), $\frac{dy}{dt} = 0$

$$\therefore \quad 0 = -\omega^2 A^2 + C \quad \text{or} \quad C = \omega^2 A^2$$

Substituting for C, we get

$$\left(\frac{dy}{dt}\right)^2 = -\omega^2 y^2 + \omega^2 A^2$$

or
$$\frac{dy}{dt} = \omega\sqrt{(A^2 - y^2)}$$

or
$$\frac{dy}{\sqrt{A^2 - y^2}} = \omega\, dt$$

Integrating both sides, we get

$$\int \frac{dy}{\sqrt{A^2 - y^2}} = \int \omega\, dt$$

where 'ϕ' is the constant of integration.

$$\sin^{-1}\left(\frac{y}{A}\right) = \omega t + \phi$$

$$\frac{y}{A} = \sin(\omega t + \phi)$$

or
$$y = A\sin(\omega t + \phi) \quad ...(4)$$

Equation (4) gives the instantaneous displacement of the body vibrating in simple harmonic motion and is the equation which represents the motion. In this equation, 'A'

represents the amplitude while 'ϕ' which is the constant of integration represents the initial phase of vibration. 'ϕ' gives us the information regarding the state of particle when we started measuring time.

(*i*) If we start measuring time from the instant the particle leaves the mean position in positive direction

$$\phi = 0$$

Therefore, equation of S.H.M. can be written as

$$y = A \sin \omega t$$

(*ii*) If the time is measured from the instant, the particle is on the positive extremity, $\phi = \pi/2$

$$\therefore \quad y = A \sin (\omega t + \pi/2)$$

or

$$y = A \cos \omega t$$

(*iii*) If the time is measured from the instant, the particle leaves the mean position in negative direction, $\phi = \pi$

$$\therefore \quad y = A \sin (\omega t + \pi)$$

or

$$y = -A \sin \omega t$$

(*iv*) If the time is measured from the instant, the particle is on the negative extremity, $\phi = 3\pi/2$

$$\therefore \quad y = A \sin (\omega t + 3\pi/2)$$

or

$$y = -A \cos \omega t.$$

5.5 ENERGY IN SIMPLE HARMONIC MOTION

A particle, executing simple harmonic motion, vibrates to and fro about its mean position. Total energy 'E' of the particle, at any instant of time, is composed of two types of energies.

(*i*) **Kinetic energy (E_k)**. *It is the energy possessed by the particle by virtue of its motion.*

If 'υ' is the instantaneous velocity of the particle, its kinetic energy 'E_k' is given by

$$E_k = \frac{1}{2} m\upsilon^2 = \frac{1}{2} m \left(\omega \sqrt{A^2 - y^2} \right)^2$$

where 'y' is the instantaneous displacement and 'A' is its amplitude.

$$E_k = \frac{1}{2} m\omega^2 (A^2 - y^2) \qquad ...(41)$$

Since velocity of the particle is maximum in the mean position while it is zero at the extreme position, its kinetic energy is also maximum in mean position while it is zero at the extreme position.

(*ii*) **Potential energy (E_p)**. *It is the energy possessed by the particle by virtue of its position.*

As the particle moves away from the mean position, restoring force is always directed towards the mean position. So, work has to be done to take it away. This work done, in removing the particle to a position away from the mean position, is called its potential energy. It can be calculated as follows.

Let 'A' be the position of the particle of mass 'm', at any instant of time, when its displacement is 'x' [Fig. 5.7].

Acceleration of the particle at $A = -\omega^2 x$

Force on the particle at A, $\vec{F} = -m\omega^2 x$

Negative sign indicates that it is directed towards the mean position.

If 'dW' is the work done in displacing it through a small distance 'dx' from A to B.

$$dW = \vec{F} . \vec{dx} = F dx \cos 180°$$

$$= (-m\omega^2 x)\, dx\, (-1)$$

or

$$dW = m\omega^2 x dx$$

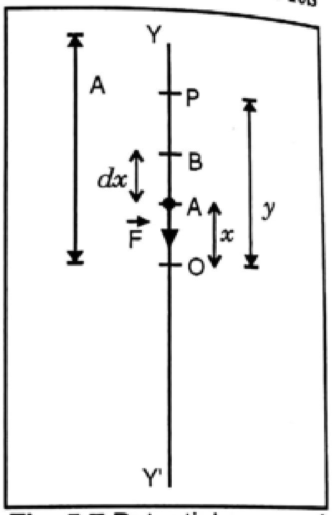


Fig. 5.7 Potential energy of the particle at a distance 'y' from mean position.

Work done 'W' in removing the particle from O to P can be obtained by integrating it between the limits 0 to y. This will be the potential energy 'E_p' of the particle at any instant.

$$E_p = W = \int_0^W dW \qquad \therefore \qquad = \int_0^y m\omega^2 x dx$$

$$E_p = m\omega^2 \int_0^y x\, dx = m\omega^2 \left[\frac{x^2}{2} \right]_0^y$$

$$E_p = m\omega^2 \left(\frac{y^2}{2} - \frac{0}{2} \right)$$

or

$$E_p = \frac{1}{2} m\omega^2 y^2 \qquad ...(15)$$

Total energy (E)

Total energy 'E' of the particle, at any instant of time, is the sum total of instantaneous kinetic and potential energies.

$$E = E_k + E_p = \frac{1}{2} m\omega^2 (A^2 - y^2) + \frac{1}{2} m\omega^2 y^2$$

$$= \frac{1}{2} m\omega^2 (A^2 - y^2 + y^2)$$

$$= \frac{1}{2} \mathbf{m\omega^2 A^2} \quad ...(16)$$

From equation (16), it is clear that the instantaneous total energy is independent of the displacement of the particle, *i.e.*, whatever, the displacement may be it is going to remain the same. In other words *total energy of a particle executing simple harmonic motion always remains constant.* Kinetic energy E_k, potential energy 'E_p', and total energy 'E' plotted as a function of time are shown in Fig. 5.8. The curves indicate that 'E_k' and 'E_p' are complementary to each other. As the particle moves away from mean position, kinetic energy gets transformed into potential energy till the transformation is complete at extreme position. While coming from extreme to mean position it gets converted from potential to kinetic energy. This conversion is in accordance with the law of conservation of energy.

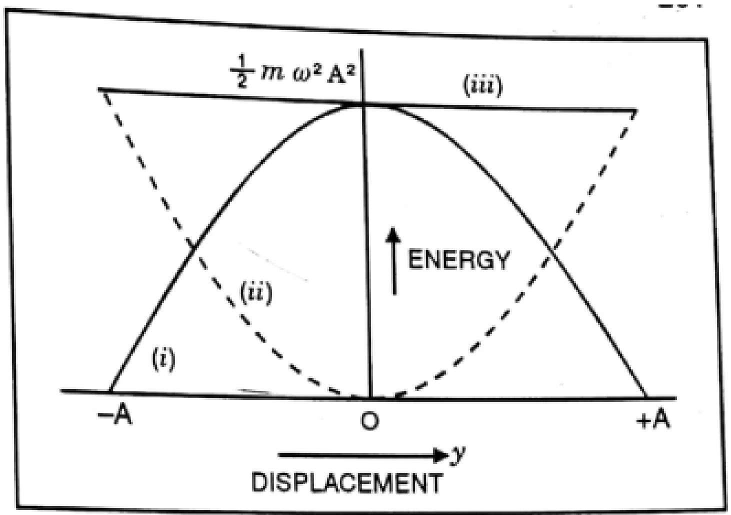


Fig. 5.8 Energy versus displacement : (*i*) Kinetic energy ; (*ii*) Potential energy ; (*iii*) Total energy.

5.7 AVERAGE VALUES OF KINETIC AND POTENTIAL ENERGY

It can be shown that in simple harmonic motion the average values of kinetic and potential energies over a complete time period are same and that is equal to half the total energy of the particle.

(*i*) **Average kinetic energy.** Kinetic energy of the particle, in S.H.M. is given by

$$K.E. = \frac{1}{2} m\, \omega^2 (A^2 - y^2)$$

Since

$$y = A \sin(\omega t + \phi)$$

$$K.E. = \frac{1}{2} m\, \omega^2 \left[A^2 - A^2 \sin^2(\omega t - \phi) \right]$$

$$= \frac{1}{2} m\, \omega^2 A^2 \left[1 - \sin^2(\omega t + \phi) \right] = \frac{1}{2} m\, \omega^2 A^2 \cos^2(\omega t + \phi)$$

But

$$\cos^2(\omega t + \phi) = \frac{1 + \cos 2(\omega t + \phi)}{2}$$

$$\therefore \quad K.E. = \frac{1}{2} m\, \omega^2 A^2 \left[\frac{1 + \cos 2(\omega t + \phi)}{2} \right]$$

$$\therefore \quad (K.E.)_{av} = \frac{1}{T} \int_0^T \frac{1}{2} m\, \omega^2 A^2 \left[\frac{1 + \cos 2(\omega t + \phi)}{2} \right] dt$$

$$= \frac{1}{4T} m\,\omega^2 A^2 \left[\int_0^T dt + \int_0^T \cos 2(\omega t + \phi)\, dt \right]$$

$$(K.E.)_{av} = \frac{1}{4T} m\,\omega^2 A^2 \left[t + \frac{\sin 2\,(\omega t + \phi)}{2} \right]_0^T$$

$$= \frac{1}{4T} m\,\omega^2 A^2 \left[\left(T + \frac{\sin 2\,(\omega T + \phi)}{2} \right) - \left(0 + \frac{\sin 2\,\phi}{2} \right) \right]$$

$$= \frac{1}{4T} m\,\omega^2 A^2 \left[T + \frac{\sin(2\,\omega T + 2\phi)}{2} - \frac{\sin 2\phi}{2} \right]$$

$$= \frac{1}{4T} m\,\omega^2 A^2 \left[T + \frac{\sin 2\,\phi}{2} - \frac{\sin 2\phi}{2} \right]$$

$$(K.E.)_{av} = \frac{1}{4} m\,\omega^2 A^2 \qquad ...(17)$$

(ii) **Average potential energy.** Instantaneous potential energy of the particle vibrating in S.H.M. is given by

$$P.E. = \frac{1}{2} m\,\omega^2 y^2$$

Since $y = A \sin\,(\omega t + \phi)$

$$\therefore \quad P.E. = \frac{1}{2} m\,\omega^2 A^2 \sin^2\,(\omega t + \phi)$$

But $\sin^2\,(\omega t + \phi) = \dfrac{1 - \cos 2\,(\omega t + \phi)}{2}$

$$\therefore \quad P.E. = \frac{1}{2} m\,\omega^2 A^2 \left[\frac{1 - \cos 2\,(\omega t + \phi)}{2} \right]$$

$$\therefore \quad (P.E.)_{av} = \frac{1}{T} \int_0^T \frac{1}{2} m\,\omega^2 A^2 \left[\frac{1 - \cos 2\,(\omega t + \phi)}{2} \right] dt$$

$$= \frac{1}{4T} \times m\,\omega^2 A^2 \left[\int_0^T dt - \int_0^T \cos 2\,(\omega t + \phi)\, dt \right]$$

$$= \frac{1}{4T} \times m\,\omega^2 A^2 \left[t - \frac{\sin 2\,(\omega t + \phi)}{2} \right]_0^T$$

$$(P.E.)_{av} = \frac{1}{4T} m\,\omega^2 A^2 \left[\left(T - \frac{\sin 2\,(\omega T + \phi)}{2} \right) - \left(0 - \frac{\sin 2\,\phi}{2} \right) \right]$$

$$= \frac{1}{4T} m\,\omega^2 A^2 \left[T - \frac{\sin 2\,\phi}{2} + \frac{\sin 2\,\phi}{2} \right] \quad [\because 2\omega T = 4\pi]$$

$$= \frac{1}{4T} m\omega^2 A^2 \times T$$

$$(P.E.)_{av} = \frac{1}{4} m\,\omega^2 A^2 \qquad ...(18)$$

From equations (17) and (18)

$$(K.E.)_{av} = (P.E.)_{av} = \frac{1}{4} m\,\omega^2 A^2 = \frac{1}{2} \times \frac{1}{2} m\,\omega^2 A^2 = \frac{1}{2} \times \text{total energy}$$

Thus, average value of kinetic and potential energy is same and is equal to half the total energy of the particle.

Free vibrations:

When a body is allowed to vibrate freely, it vibrates with a definite frequency. Such a vibration is called free vibration and the frequency with which the body vibrates is called its natural frequency of vibration. Its frequency depends on its size, mass, the elasticity of its material and the local gravity.

A free vibration is a SHM. If a body of mass 'm' executes SHM along y-direction w.r.t. time 't', then the restoring force,

$$F \propto -y$$
$$\Rightarrow F = -ky$$

Where k is the constant of proportionality.

Thus
$$\Rightarrow m\frac{d^2y}{dt^2} = -ky$$
$$\Rightarrow \frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$
$$\Rightarrow \frac{d^2y}{dt^2} + \omega^2 y = 0 \qquad ....(1)$$

Where $\omega^2 = \frac{k}{m} =$ the restoring force constant for unit deviation for unit mass = Spring constant for unit mass.

The equation (1) is called the equation of free vibration of a body.

Damped vibrations:

If the amplitude of vibration of a body decreases with time due to the presence of external frictional force (air, viscous) then the vibration is called damped vibration.

Suppose a particle of mass 'm' is oscillating along y-axis under the action of a restoring force proportional to the displacement 'y' from the position of equilibrium (i.e. $= -\mu y$) and a damping force proportional to the instantaneous velocity (i.e. $= -R\frac{dy}{dt}$).These two force are balanced by the inertial force of the body, i.e.

$$F = -\mu y - R\frac{dy}{dt}$$
$$\Rightarrow m\frac{d^2y}{dt^2} = -\mu y - R\frac{dy}{dt}$$
$$\Rightarrow \frac{d^2y}{dt^2} + \frac{R}{m}\frac{dy}{dt} + \frac{\mu}{m}y = 0$$
$$\Rightarrow \frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega^2 y = 0 \qquad \text{----- (1)}$$

Where μ is a constant and is called the spring factor or stiffness factor, R is a frictional force per unit velocity, $2b = \frac{R}{m}$ is the resisting force per unit mass per unit velocity or coefficient of friction i.e. damping coefficient and $\omega^2 = \frac{\mu}{m}$ is the resisting force per unit displacement per unit mass or angular frequency of vibration in the absence of damping.

To solve the equation, let $y = ce^{\alpha t}$ be a trial solution, where 'c' and α are arbitrary constants. Differentiating we get

$$\frac{dy}{dt} = \alpha ce^{\alpha t} \text{ and } \frac{d^2y}{dt^2} = \alpha^2 ce^{\alpha t}$$

Therefore, from equation (1), we get

$$(\alpha^2 + 2b\alpha + \omega^2)ce^{\alpha t} = 0$$
$$\Rightarrow (\alpha^2 + 2b\alpha + \omega^2) = 0$$

$\because$ *the solution of the equation* $ax^2 + bx + c = 0$ *is* $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

It has two roots,

$$\alpha_1 = \frac{-2b + \sqrt{(2b)^2 - 4\omega^2}}{2} \quad \text{and} \quad \alpha_2 = \frac{-2b - \sqrt{(2b)^2 - 4\omega^2}}{2}$$

$$\Rightarrow \alpha_1 = \frac{2\left(-b + \sqrt{b^2 - \omega^2}\right)}{2} \qquad \Rightarrow \alpha_2 = \frac{2\left(-b - \sqrt{b^2 - \omega^2}\right)}{2}$$

$$\Rightarrow \alpha_1 = -b + \sqrt{b^2 - \omega^2} \qquad \Rightarrow \alpha_2 = -b - \sqrt{b^2 - \omega^2}$$

Therefore, the two possible solutions of equation (1) are

$$y_1 = c_1 e^{\alpha_1 t} \quad \text{and} \quad y_2 = c_2 e^{\alpha_2 t}$$

Since the equation (1) is a linear homogeneous equation its general solution is given by the principle of superposition as

$$y = y_1 + y_2$$

$$\Rightarrow y = c_1 e^{\alpha_1 t} + c_2 e^{\alpha_2 t}$$

$$\Rightarrow y = c_1 e^{\left(-b+\sqrt{b^2-\omega^2}\right)t} + c_2 e^{\left(-b-\sqrt{b^2-\omega^2}\right)t}$$

$$\Rightarrow y = e^{-bt}\left[c_1 e^{\left(\sqrt{b^2-\omega^2}\right)t} + c_2 e^{\left(-\sqrt{b^2-\omega^2}\right)t}\right] \qquad \text{---- (2)}$$

Where C_1 and C_2 are two arbitrary constants to be evaluated from the given initial conditions . The nature of the solution (2) depends on the relative values of 'b' and 'ω'.

In the expression (2), both $c_1 e^{\left(-b+\sqrt{b^2-\omega^2}\right)t}$ and $c_2 e^{\left(-b-\sqrt{b^2-\omega^2}\right)t}$ tends to zero when time 't' tends to infinity. Hence $y \to 0 \quad as \quad t \to \infty$.

Let the system have a maximum displacement y_0 and zero velocity initially i.e. at $t = 0$, $y = y_0$ and $\frac{dy}{dt} = 0$.

Therefore from equation (2) we get

$$y_0 = e^0\left[c_1 e^0 + c_2 e^0\right]$$

$$\Rightarrow c_1 + c_2 = y_0 \qquad \text{---------- (3)}$$

And $$\frac{dy}{dt} = \frac{d}{dt}\left[c_1 e^{\left(-b+\sqrt{b^2-\omega^2}\right)t} + c_2 e^{\left(-b-\sqrt{b^2-\omega^2}\right)t}\right]$$

$$= c_1\left(-b + \sqrt{b^2 - \omega^2}\right) e^{\left(-b+\sqrt{b^2-\omega^2}\right)t} + c_2\left(-b - \sqrt{b^2 - \omega^2}\right) e^{\left(-b-\sqrt{b^2-\omega^2}\right)t}$$

At $t = 0$, $\frac{dy}{dt} = 0$.

$$\Rightarrow 0 = c_1\left(-b + \sqrt{b^2 - \omega^2}\right)e^0 + c_2\left(-b - \sqrt{b^2 - \omega^2}\right)e^0$$

$$\Rightarrow 0 = c_1\left(-b + \sqrt{b^2 - \omega^2}\right) + c_2\left(-b - \sqrt{b^2 - \omega^2}\right)$$

$$\Rightarrow -b(c_1 + c_2) + \sqrt{b^2 - \omega^2}\,(c_1 - c_2) = 0$$

$$\Rightarrow -b\,y_0 + \sqrt{b^2-\omega^2}\left(c_1-c_2\right)=0 \quad \text{(using equation 3)}$$

$$\Rightarrow \left(c_1-c_2\right)=\frac{b\,y_0}{\sqrt{b^2-\omega^2}} \qquad \text{-------- (4)}$$

Now equation $(3)+(4)\Rightarrow$

$$\Rightarrow c_1+c_2+c_1-c_2 = y_0+\frac{b\,y_0}{\sqrt{b^2-\omega^2}}$$

$$\Rightarrow 2c_1 = y_0\left(1+\frac{b}{\sqrt{b^2-\omega^2}}\right)$$

$$\Rightarrow c_1 = \frac{1}{2}y_0\left(1+\frac{b}{\sqrt{b^2-\omega^2}}\right) \qquad \text{------- (5)}$$

And equation $(3)-(4)\Rightarrow$

$$\Rightarrow c_1+c_2-c_1+c_2 = y_0-\frac{b\,y_0}{\sqrt{b^2-\omega^2}}$$

$$\Rightarrow c_2 = \frac{1}{2}y_0\left(1-\frac{b}{\sqrt{b^2-\omega^2}}\right) \qquad \text{------ (6)}$$

Substituting the values of C_1 and C_2 in equation (2), we get

$$\Rightarrow y = \frac{1}{2}y_0\left(1+\frac{b}{\sqrt{b^2-\omega^2}}\right)e^{\left(-b+\sqrt{b^2-\omega^2}\right)t}+\frac{1}{2}y_0\left(1-\frac{b}{\sqrt{b^2-\omega^2}}\right)e^{\left(-b-\sqrt{b^2-\omega^2}\right)t}$$

Case-I:Overdamped motion i.e. Large damping: If the damping force is very large i.e. $b^2 >> \omega^2$ then $\sqrt{b^2-\omega^2}$ is real (positive quantity), the value of 'y' consists of two terms both together falling exponentially to zero and the motion is non-oscillatory, aperiodic or dead beat type. If the motion is started with an initial displacement 'y_0' but no initial velocity, then the displacement gradually falls off to zero with time and the body returns to the equilibrium position without any oscillation about the equilibrium position. This type of motion is found in a dead beat galvanometer or a pendulum immersed in a highly viscous liquid. The variation of displacement 'y' with time't' is shown in fig. 1-I.

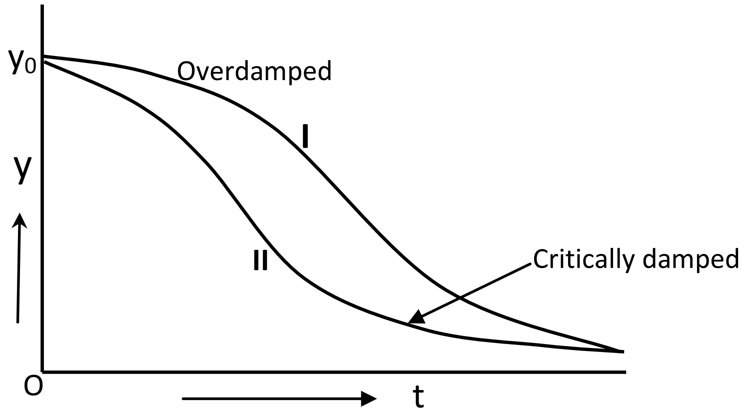


Fig. 1-I and II

Case-II: Critical damping: If $b^2=\omega^2$ then the motion is also non-oscillatory, but here the rate of decay is much faster than the overdamped case and the motion is called critical damping. The variation of displacement 'y' with time't' is shown in fig.1-II.

The critically damped condition is useful if we desire most quick decay without oscillation. In a pointer-type galvanometer we want the pointer to move smoothly and quickly to a new equilibrium position when current is sent through it. This helps us to take the reading immediately after the meter is connected to a circuit.

Case-III: Damped oscillation i.e. Small damping: If the damping force is small i.e. $b^2 < \omega^2$ then $\sqrt{b^2-\omega^2}$ is negative quantity i.e. imaginary and we can write

$$\sqrt{b^2-\omega^2} = \sqrt{-\left(\omega^2-b^2\right)} = \sqrt{-1}\sqrt{\left(\omega^2-b^2\right)} = i\sqrt{\omega^2-b^2}$$

Hence equation (2) reduces to

$$y = e^{-bt}\left[c_1 e^{\left(\sqrt{b^2-\omega^2}\right)t} + c_2 e^{\left(-\sqrt{b^2-\omega^2}\right)t}\right]$$

$$\Rightarrow y = e^{-bt}\left[c_1 e^{\left(i\sqrt{\omega^2-b^2}\right)t} + c_2 e^{\left(-i\sqrt{\omega^2-b^2}\right)t}\right]$$

$\because \quad e^{ix} = \cos x + i\sin x$ and $e^{-ix} = \cos x - i\sin x$

$$\therefore y = e^{-bt}\left[c_1\cos\left(\sqrt{\omega^2-b^2}\right)t + ic_1\sin\left(\sqrt{\omega^2-b^2}\right)t + c_2\cos\left(\sqrt{\omega^2-b^2}\right)t - ic_2\sin\left(\sqrt{\omega^2-b^2}\right)t\right]$$

$$\Rightarrow y = e^{-bt}\left[(c_1+c_2)\cos\left(\sqrt{\omega^2-b^2}\right)t + i(c_1-c_2)\sin\left(\sqrt{\omega^2-b^2}\right)t\right]$$

Let $(c_1+c_2) = A_1$, $\quad i(c_1-c_2) = A_2 \quad$ *and* $\quad \sqrt{\omega^2-b^2} = \omega_0$

$$\therefore y = e^{-bt}[A_1\cos\omega_0 t + A_2\sin\omega_0 t]$$

Putting $A_1 = A_0\cos\phi$ *and* $A_2 = A_0\sin\phi$, we can write

$$y = e^{-bt}[A_0\cos\omega_0 t\cos\phi + A_0\sin\omega_0 t\sin\phi]$$

$$\Rightarrow y = A_0 e^{-bt}[\cos\omega_0 t\cos\phi + \sin\omega_0 t\sin\phi]$$

$$\Rightarrow y = A_0 e^{-bt}\cos(\omega_0 t - \phi) \qquad \text{------- (7)}$$

Where 'A_0' and 'ϕ' are real constant to be evaluated from the given initial condition . Equation (7) represents a damped oscillatory motion with an angular frequency $\omega_0 = \sqrt{\omega^2-b^2}$, whose amplitude A_0e^{-bt} decreases exponentially with time as shown in fig.2. and due to the presence of damping the frequency slightly reduces i.e. the time period is slightly greater than the time period for free natural vibration.

To evaluate the constant 'A_0' and 'ϕ', let at $t=0$, $y = y_0$ *and* $\frac{dy}{dt} = v_0$.

Therefore, from equation (7),

$$y_0 = A_0\cos\phi \qquad \text{------- (8)}$$

And $$\frac{dy}{dt} = \frac{d}{dt}\left[A_0 e^{-bt}\cos(\omega_0 t - \phi)\right]$$

$$= -A_0 b e^{-bt}\cos(\omega_0 t - \phi) - A_0\omega_0 e^{-bt}\sin(\omega_0 t - \phi)$$

$$\therefore v_0 = -A_0 b\cos\phi + A_0\omega_0\sin\phi$$

$$\Rightarrow v_0 = -y_0 b + A_0\omega_0\sin\phi$$

$$\Rightarrow A_0\sin\phi = \frac{v_0 + y_0 b}{\omega_0} \qquad \text{--------- (9)}$$

Combining equations (8) and (9), we get

$$A_0 = \sqrt{{y_o}^2 + \left(\frac{v_0 + y_0 b}{\omega_0}\right)^2}$$

And $\tan\phi = \dfrac{v_0 + y_0 b}{y_0 \omega_0}$

If at $t = 0, \quad y = y_0 \quad and \quad \dfrac{dy}{dt} = v_0 = 0$, then

$$A_0 = \sqrt{{y_o}^2 + \left(\frac{y_0 b}{\omega_0}\right)^2} = y_0\sqrt{1 + \frac{b^2}{{\omega_0}^2}} = y_0\sqrt{1 + \frac{b^2}{\omega^2 - b^2}} = y_0\sqrt{\frac{\omega^2 - b^2 + b^2}{\omega^2 - b^2}}$$

$$\Rightarrow A_0 = y_0\sqrt{\frac{\omega^2}{\omega^2 - b^2}} = \frac{y_0\omega}{\sqrt{\omega^2 - b^2}}$$

And $\tan\phi = \dfrac{y_0 b}{y_0\omega_0} = \dfrac{b}{\omega_0} = \dfrac{b}{\sqrt{\omega^2 - b^2}}$

Therefore, from equation (7), we get

$$y = A_0 e^{-bt}\cos(\omega_0 t - \phi)$$

$$\Rightarrow y = \frac{y_0\omega}{\sqrt{\omega^2 - b^2}} e^{-bt}\cos\left(\sqrt{\omega^2 - b^2}\, t - \tan^{-1}\frac{b}{\sqrt{\omega^2 - b^2}}\right) \qquad \text{---(10)}$$

The motion expressed by equation (10) is a damped oscillatory motion, the amplitude decreasing exponentially with time.

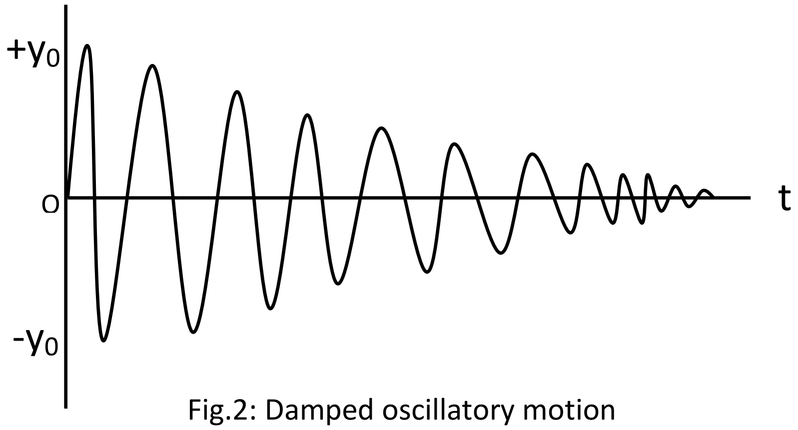


Fig.2: Damped oscillatory motion

Logarithmic decrement:

In case of damped oscillations, the amplitude of the oscillations decreases exponentially with time by damping factor e^{-bt}. This decrease in amplitude is often expressed in terms of the logarithmic decrement 'λ', which is defined as the natural logarithmic of the ratio between the two successive amplitudes on the same side of the mean position of a damped oscillation separated by one full period.

The amplitude of the damped oscillations at any time 't' is given by

$$A = A_0 e^{-bt}$$

If the oscillation starts from mean position, then after a time $t = \dfrac{T}{4}$ where 'T' is the time period, the oscillating particle goes to extreme position. Let the first amplitude be denoted A_1, then

$$A_1 = A_0 e^{-\frac{bT}{4}}$$

The particle will come to the mean position and then go to the extreme position on the other side, again come back to the mean position and go to the extreme position on the same side after a time T, i.e. $T+\frac{T}{4}$ from start.

Let the second amplitude be A_2, then

$$A_2 = A_0 e^{-b\left(T+\frac{T}{4}\right)}$$

Similarly, the successive amplitudes of the 3rd, 4th, 5th, etc. oscillations will be given by

$$A_3 = A_0 e^{-b\left(2T+\frac{T}{4}\right)}$$

$$A_4 = A_0 e^{-b\left(3T+\frac{T}{4}\right)}$$

$$A_n = A_0 e^{-b\left((n-1)T+\frac{T}{4}\right)}$$

Hence $$\frac{A_1}{A_2} = \frac{A_2}{A_3} = \frac{A_3}{A_4} = \ldots\ldots\ldots = \frac{A_{n-1}}{A_n} = e^{bT}$$

Taking natural logarithms (to the base e), we have

$$\log_e \frac{A_1}{A_2} = \log_e \frac{A_2}{A_3} = \log_e \frac{A_3}{A_4} = \ldots\ldots\ldots = \log_e \frac{A_{n-1}}{A_n} = bT = \lambda \quad (say)$$

Where 'λ' is the logarithmic decrement.

Force Vibration:

The oscillation produced by an oscillator under the effect of an external periodic force of frequency other than the natural frequency of the oscillator is called force vibration.

Let an external periodic force $F \sin pt$ of frequency $\frac{p}{2\pi}$ and amplitude F is acting on a particle of mass 'm' executing S.H.M. in a resisting medium. If 'y' be the displacement of the particle then the equation of motion of the particle is

$$m\frac{d^2y}{dt^2} = -\mu y - R\frac{dy}{dt} + F\sin pt$$

$$\Rightarrow m\frac{d^2y}{dt^2} + R\frac{dy}{dt} + \mu y = F\sin pt$$

$$\Rightarrow \frac{d^2y}{dt^2} + \frac{R}{m}\frac{dy}{dt} + \frac{\mu}{m}y = \frac{F}{m}\sin pt$$

$$\Rightarrow \frac{d^2y}{dt^2} + k\frac{dy}{dt} + \omega^2 y = f\sin pt \qquad \text{---------} (1)$$

Where $k = \frac{R}{m}$ is the damping coefficient, $\omega^2 = \frac{\mu}{m}$, μ is the stiffness constant or spring constant, $\frac{\omega}{2\pi}$ is the natural frequency of the system and $f = \frac{F}{m}$.

Now, replacing $\frac{d}{dt}$ by 'D' in equation (1), we get

$$D^2y + kDy + \omega^2 y = f\sin pt$$
$$\Rightarrow \left(D^2 + kD + \omega^2\right)y = f\sin pt$$
$$\Rightarrow y = \frac{f\sin pt}{\left(D^2 + kD + \omega^2\right)}$$
$$\Rightarrow y = \frac{f\sin pt}{\omega^2 - p^2 + kD} \qquad \text{Writing } -p^2 \text{ for } D^2$$
$$\Rightarrow y = \frac{f\left[\left(\omega^2 - p^2\right) - kD\right]\sin pt}{\left(\omega^2 - p^2\right)^2 - k^2D^2}$$
$$\Rightarrow y = \frac{f\left[\left(\omega^2 - p^2\right)\sin pt - kD\sin pt\right]}{\left(\omega^2 - p^2\right)^2 + k^2p^2}$$
$$\Rightarrow y = \frac{f}{\left(\omega^2 - p^2\right)^2 + k^2p^2}\left[\left(\omega^2 - p^2\right)\sin pt - k\frac{d}{dt}\sin pt\right] \qquad \because D = \frac{d}{dt}$$
$$\Rightarrow y = \frac{f}{\left(\omega^2 - p^2\right)^2 + k^2p^2}\left[\left(\omega^2 - p^2\right)\sin pt - kp\cos pt\right]$$
$$\Rightarrow y = \frac{f}{\sqrt{\left(\omega^2 - p^2\right)^2 + k^2p^2}}\left[\frac{\left(\omega^2 - p^2\right)}{\sqrt{\left(\omega^2 - p^2\right)^2 + k^2p^2}}\sin pt - \frac{kp\cos pt}{\sqrt{\left(\omega^2 - p^2\right)^2 + k^2p^2}}\right]$$

Let $\frac{\left(\omega^2 - p^2\right)}{\sqrt{\left(\omega^2 - p^2\right)^2 + k^2p^2}} = \cos\in$ And $\frac{kp}{\sqrt{\left(\omega^2 - p^2\right)^2 + k^2p^2}} = \sin\in$

Where, $\in = \tan^{-1}\frac{kp}{\left(\omega^2 - p^2\right)}$

Therefore,

$$y = \frac{f}{\sqrt{\left(\omega^2 - p^2\right)^2 + k^2p^2}}\left[\sin pt\ \cos\in - \cos pt\ \sin\in\right] \Rightarrow y = \frac{F}{m\sqrt{\left(\omega^2 - p^2\right)^2 + k^2p^2}}\sin(pt - \in)$$
$$\Rightarrow y = \frac{F}{pZ}\sin(pt - \in) \qquad \text{------- (2)}$$

Here, $Z = \frac{m}{p}\sqrt{\left(\omega^2 - p^2\right)^2 + k^2p^2}$

$$\Rightarrow Z = \frac{m}{p}\sqrt{\frac{p^2\left(\omega^2 - p^2\right)^2}{p^2} + k^2p^2}$$
$$\Rightarrow Z = \frac{m}{p}p\sqrt{\left\{\frac{\left(\omega^2 - p^2\right)}{p}\right\}^2 + k^2}$$
$$\Rightarrow Z = m\sqrt{\left(\frac{\omega^2}{p} - p\right)^2 + k^2}$$
$$\Rightarrow Z = \sqrt{m^2\left(\frac{\omega^2}{p} - p\right)^2 + k^2m^2} \qquad \because k = \frac{R}{m} \Rightarrow R = km$$
$$\Rightarrow Z = \sqrt{\left(\frac{m\omega^2}{p} - mp\right)^2 + R^2}$$
$$\Rightarrow Z = \sqrt{\left(\frac{m\mu}{pm} - mp\right)^2 + R^2} \qquad \because \omega^2 = \frac{\mu}{m},$$

$$\Rightarrow Z = \sqrt{\left(\frac{\mu}{p} - mp\right)^2 + R^2}$$

$$\Rightarrow Z = \sqrt{X^2 + R^2} \qquad \text{----------- (3)}$$

Where $X = \frac{\mu}{p} - mp$

Hence $y = \frac{F}{p\sqrt{X^2 + R^2}}\sin(pt - \in)$ ------------ (4)

Where Z is known as mechanical impedance given by $Z = \sqrt{X^2 + R^2}$, X is the Reactance and R is the resistance.

(i) If μ is large then Z reduces to $\frac{\mu}{p}$ and the system is said to be stiffness controlled.

(ii) If R is large then Z reduces to R and the system is said to be resistance controlled.

(iii) If m is large then Z becomes equal to mp and the system is said to be mass controlled.

Amplitude Resonance: Maximum Displacement of the system:

The displacement of force vibration is

$$y = \frac{F}{pZ}\sin(pt - \in)$$

$$\because \quad Z = \frac{m}{p}\sqrt{(\omega^2 - p^2)^2 + k^2p^2}$$

$$\therefore \quad y = \frac{F}{m\sqrt{(\omega^2 - p^2)^2 + k^2p^2}}\sin(pt - \in) \qquad \text{----- (1)}$$

Displaced amplitude is maximum when the values of m or $\sqrt{(\omega^2 - p^2)^2 + k^2p^2}$ in equation (1) is minimum i.e. for maximum displacement,

$$\frac{d}{dt}\left(\sqrt{(\omega^2 - p^2)^2 + k^2p^2}\right) = 0$$

$$\Rightarrow \frac{1}{2}\left((\omega^2 - p^2)^2 + k^2p^2\right)^{-1/2}\frac{d}{dt}\left((\omega^2 - p^2)^2 + k^2p^2\right) = 0$$

$$\Rightarrow \frac{1}{2\left(\sqrt{(\omega^2 - p^2)^2 + k^2p^2}\right)} 2(\omega^2 - p^2) - 2p + 2pk^2 = 0$$

$$\Rightarrow -4p(\omega^2 - p^2) + 2pk^2 = 0$$

$$\Rightarrow 4p(\omega^2 - p^2) = 2pk^2$$

$$\Rightarrow (\omega^2 - p^2) = \frac{k^2}{2}$$

$$\Rightarrow p^2 = \omega^2 - \frac{k^2}{2}$$

$$\Rightarrow p = \sqrt{\omega^2 - \frac{k^2}{2}} \qquad \text{------- (2)}$$

Which is the condition for amplitude resonance .

Thus displacement amplitude is maximum when forcing frequency $\frac{p}{2\pi}$ is equal to $\frac{1}{2\pi}\sqrt{\omega^2 - \frac{k^2}{2}}$ that is when the forcing frequency is slightly lower than natural frequency $\frac{\omega}{2\pi}$ which is nearly equal to the frequency of the damped system $\frac{1}{2\pi}\sqrt{\omega^2 - \frac{k^2}{2}}$. Therefore at $p \simeq \omega$, we have from equation (1)

$$A = \frac{F}{m\sqrt{\left(\omega^2 - p^2\right)^2 + k^2 p^2}}$$

$$\therefore \quad A = A_{max} = \frac{F}{m\sqrt{k^2\omega^2}}$$

$$= \frac{F}{mk\omega} = \frac{f}{k\omega} \qquad \because \frac{F}{m} = f$$

Fig. 1 represents the curves showing the variation of amplitude with the change of frequency of the force with varying degree of damping. The amplitude is infinite which does not occur in the nature as 'k' is never zero. For finite value of 'k' the amplitude is maximum when p = ω.

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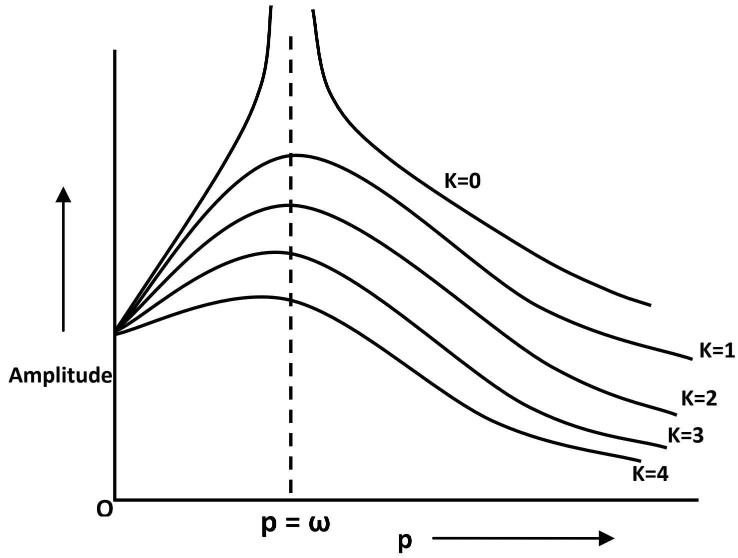

Fig. 1