Elasticity

Elasticity is the property by virtue of which material bodies regain their original shape and size after the external deforming forces are removed. When an external force act on a body, there is change in its length, shape and volume. The body is saying to be strained.

When this external force is removed, the body regains its original shape and size. Such bodies are called elastic bodies. Steel, glass ivory, quartz etc. are elastic bodies. The bodies which do not regain their original shape and size are called plastic bodies. The opposite of elasticity is plasticity. No substance is perfectly elastic or perfectly (completely) plastic. Steel is more elastic than rubber. Liquid and gases are highly elastic.

Stress: It is defined as the restoring force per unit area set up in the body, when deformed by the external force. Thus

$$Stress = \frac{Restoring\ force}{Area} = \frac{External\ force\ applied}{Area} = \frac{F}{A}$$

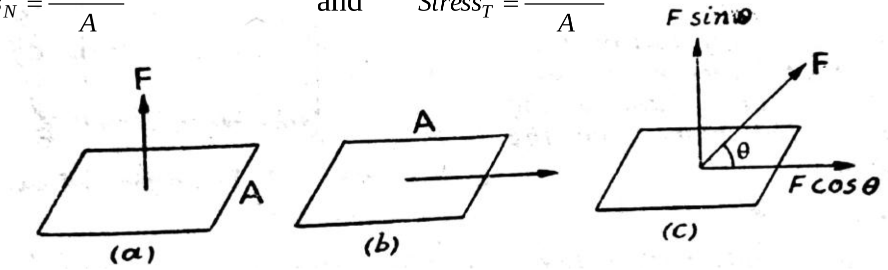
A stress can be classified into two categories.

Normal Stress: Stress is said to be normal stress if the restoring force acts at right angles to the surface as shown in fig.1(a).

Tangential stress: Stress is said to be tangential if it acts in a direction parallel to the surface as shown in fig.1(b).

Stress in any other direction can be resolved into two rectangular components as shown in fig.1(c).

$$Stress_N = \frac{F\sin\theta}{A} \quad \text{and} \quad Stress_T = \frac{F\cos\theta}{A}$$



Strain: Relative change in configuration due to the application of a deforming force is called strain i.e. the ratio of change in dimension of the body to its original dimension is called strain. It has no units.

There are three types of strain-

(a) Longitudinal strain: It is defined as the ratio between change in length δl to the original length L i.e.

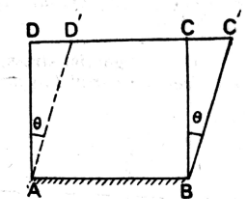
$$Longitudinal\ strain = \frac{\delta l}{L}$$

(b) Volumetric strain: It is defined as the ratio between change in volume δv to the original volume V i.e.

$$Volumetric\ strain = \frac{\delta v}{V}$$

(c) Shear strain: it is defined as the angle θ through which a line originally perpendicular to the fixed face gets turn on applying tangential deforming force.

$$Shear\ strain = \theta = \tan\theta = \frac{DD'}{AD}$$



Hooke's Law: It states that, within elastic limits, stress is proportional to strain i.e.

$$Stress \propto strain$$

$$\Rightarrow \frac{Stress}{Strain} = \text{Constant.}$$

This constant of proportionality is called modulus of elasticity or coefficient of elasticity of the material. Its value depends upon the nature of the material of the body and the manner in which the body is deformed. There are three modulus of elasticity namely Young's modulus (Y), Bulk modulus (K) and Modulus of rigidity (η). Since, strain has no units, co-efficient of elasticity possesses units of stress i.e. dyne cm^{-2} in CGS system and Nm^{-2} in MKS.

Young's Modulus (Y): Young's modulus of elasticity is defined as the ratio between normal stress and the longitudinal strain.

$$Y = \frac{Normal\ Stress}{Longitudinal\ Strain} = \frac{F/A}{\delta l/l} = \frac{Fl}{A\delta l}$$

Bulk Modulus of elasticity (K): Bulk modulus of elasticity is defined as the ratio between normal stress and the volume strain.

$$K = \frac{Normal\ Stress}{Volume\ Strain} = \frac{F/A}{-\delta v/V} = -\frac{FV}{A\delta v}$$

If the force F act on the body perpendicular to its surface from all directions then F/A can be written as pressure P.

$$K = \frac{P}{-\delta v/V}$$

Negative sign indicates that an increase in pressure (Positive) results in a decrease in volume (δv negative). The reciprocal of the bulk modulus of a substance is termed as compressibility.

Modulus of rigidity (η): Modulus of rigidity is defined as the ratio between tangential stress to the shear strain.

All the above definitions of co-efficient of elasticity hold good for small strain so that Hooks law is applicable.

4.10. Elastic Constants and Relations between them

Young's modulus of elasticity (Y), bulk modulus of elasticity (K), modulus of rigidity (η) and Poisson's ratio (σ) of a material are called the *elastic constants* of the material. Their values depend only upon the nature of material and not upon its shape and size etc. They are connected to each other by a number of relations which can be determined as follows :

(i) Relation between Y, K and σ

Consider a cube ABCDEFGH having each side '*l*' (Fig. 4.9). Let a force 'AT' be applied normally outward to each of the six faces. Thus, each pair of faces experiences an equal tensile stress 'T'. Consider one of the lines, say CD. Tensile stress parallel to CD increases its length while the stresses acting parallel to BC and CG decrease its length.

Tensile strain along CD due to tensile stress parallel to CD

$$=\alpha T$$

Compression strain along CD due to tensile strain along BC

$$=\beta T$$

Compression strain along CD due to tensile strain along CG

$$=\beta T$$

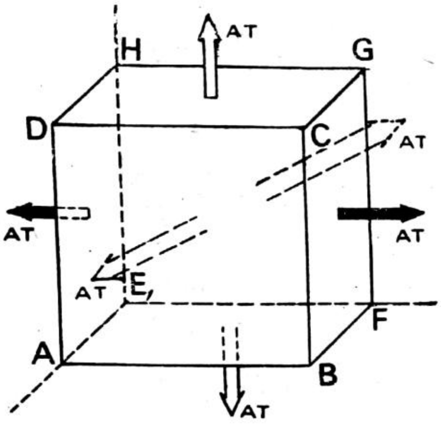


Fig. 4.9. Relation between Y, k and σ.

Therefore, net tensile strain along CD

$$=\alpha T-2\beta T$$
$$=(\alpha-2\beta)T$$

Change in length of CD

$$=l(\alpha-2\beta)T$$

∴ Final length of CD

$$=l+l(\alpha-2\beta)T$$
$$=l[1+(\alpha-2\beta)T]$$

∴ Final volume of the cube,

$$V=l^3[1+(\alpha-2\beta)T]^3$$

Applying Binomial theorem, since $(\alpha-2\beta)$ is very small

∴ $$V=l^3[1+3(\alpha-2\beta)T]$$

or $$V=V_0[1+3(\alpha-2\beta)T]$$

or $$V-V_0=3V_0(\alpha-2\beta)T$$

∴ Bulk modulus, $$K=\frac{T}{\frac{V-V_0}{V_0}}$$

or $$K=\frac{T}{\frac{3V_0(\alpha-2\beta)T}{V_0}}$$

or $$K=\frac{1}{3(\alpha-2\beta)} \quad ...(1)$$

or $$K=\frac{1}{3\alpha\left(1-2\frac{\beta}{\alpha}\right)}$$

$$=\frac{\frac{1}{\alpha}}{3\left(1-2\frac{\beta}{\alpha}\right)}$$

But $\frac{1}{\alpha}=Y$ and $\frac{\beta}{\alpha}=\sigma$

∴ $$K=\frac{Y}{3(1-2\sigma)} \quad ...(2)$$

which is the required relation.

(*ii*) Relation between Y, η and σ

Consider a section ABCD of a cube having its lower face fixed. Let a force 'F' be applied along DC (Fig. 4.10). As discussed in the previous section, shearing stress along DC is equivalent to a tensile stress along AC and a compressive stress along BD. Therefore, it results in an increase in AC and a decrease in BD.

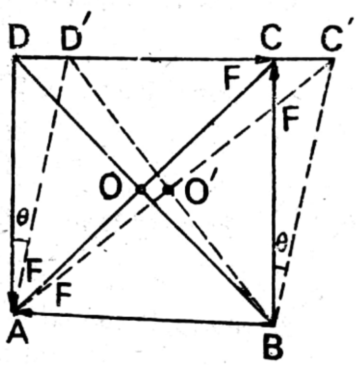


Fig. 4.10. Relation between Y, η and σ.

Let us, first of all, consider the increase in length along AC. This increase is due to two factors :

(*i*) Increase in length of AC due to tensile stress along AC $= AC.\alpha T$

(*ii*) Increase in length of AC due to compressive stress along DB $= AC.\beta T$

Net increase in length of AC

$$= AC.\alpha T + AC.\beta T$$
$$= AC(\alpha+\beta)\, T$$
$$= l\sqrt{2}\,(\alpha+\beta)\, T$$

$\therefore$ Final length of AC,

$$AC' = l\sqrt{2} + l\sqrt{2}\,(\alpha+\beta)\, T$$
$$= l\sqrt{2}[1+(\alpha+\beta)\, T]$$

Considering diagonal BD, decrease in length of BD is also due to two factors *i.e.* due to a tensile stress along AC and due to compressive stress along BD

$\therefore$ Final length of BD,

$$BD' = l\sqrt{2}\,[1-(\alpha+\beta)\, T]$$
$$\angle O'AB = \tfrac{1}{2}\angle D'AB$$
$$= \frac{1}{2}\left(\frac{\pi}{2} - \theta\right)$$
$$= \left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

where 'θ' is the shear strain produced as a result of the application of shear stress.

$$\therefore \quad \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \tan\,(\angle O'AB)$$
$$= \frac{O'B}{O'A} = \frac{BD'}{AC'}$$

Substituting for BD′ and AC′

$$\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \frac{l\sqrt{2}[1-(\alpha+\beta)\, T]}{l\sqrt{2}[1+(\alpha-\beta)\, T]}$$

or
$$\frac{\tan\frac{\pi}{4} - \tan\frac{\theta}{2}}{1+\tan\frac{\pi}{4}\tan\frac{\theta}{2}} = \frac{1-(\alpha+\beta)T}{1+(\alpha+\beta)T}$$

or
$$\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}} = \frac{1-(\alpha+\beta)T}{1+(\alpha+\beta)T}$$

Comparing the terms on two sides

$$\therefore \quad \tan\frac{\theta}{2}=(\alpha+\beta)\,T$$

If the shear strain is small,

$$\tan\frac{\theta}{2}=\frac{\theta}{2}$$

$$\therefore \quad (\alpha+\beta)T=\frac{\theta}{2}$$

$$\therefore \quad \frac{T}{\theta}=\frac{1}{2(\alpha+\beta)}$$

$$\sin\ \frac{T}{\theta}=\eta \quad \text{(Modulus of rigidity)}$$

$$\therefore \quad \eta=\frac{1}{2(\alpha+\beta)}$$

or

$$\eta=\frac{1}{2\alpha\left(1+\frac{\beta}{\alpha}\right)}=\frac{\frac{1}{\alpha}}{2\left(1+\frac{\beta}{\alpha}\right)}$$

or

$$\eta=\frac{Y}{2(1+\sigma)} \quad \ldots(3)$$

which is the required relation.

(iii) Relation between K, η and σ

From equation (2) $\quad Y=3K(1-2\sigma) \quad \ldots(4)$

From equation (3) $\quad Y=2\eta(1+\sigma) \quad \ldots(5)$

From equations (4) and (5)

$$3K(1-2\sigma)=2\eta(1+\sigma)$$

$$3K-6K\sigma=2\eta+2\eta\sigma$$

$$\sigma(6K+2\eta)=3K-2\eta$$

or

$$\sigma=\frac{3K-2\eta}{6K+2\eta} \quad (6)$$

which is the required relation.

(iv) Relation between Y, K and η

From equation (2)

$$1-2\sigma=\frac{Y}{3K} \quad \ldots(7)$$

From equation (3)

$$2+2\sigma=\frac{Y}{\eta} \quad \ldots(8)$$

Adding,

$$3=\frac{Y}{3K}+\frac{Y}{\eta}$$

Multiplying throughout by $\frac{3}{Y}$,

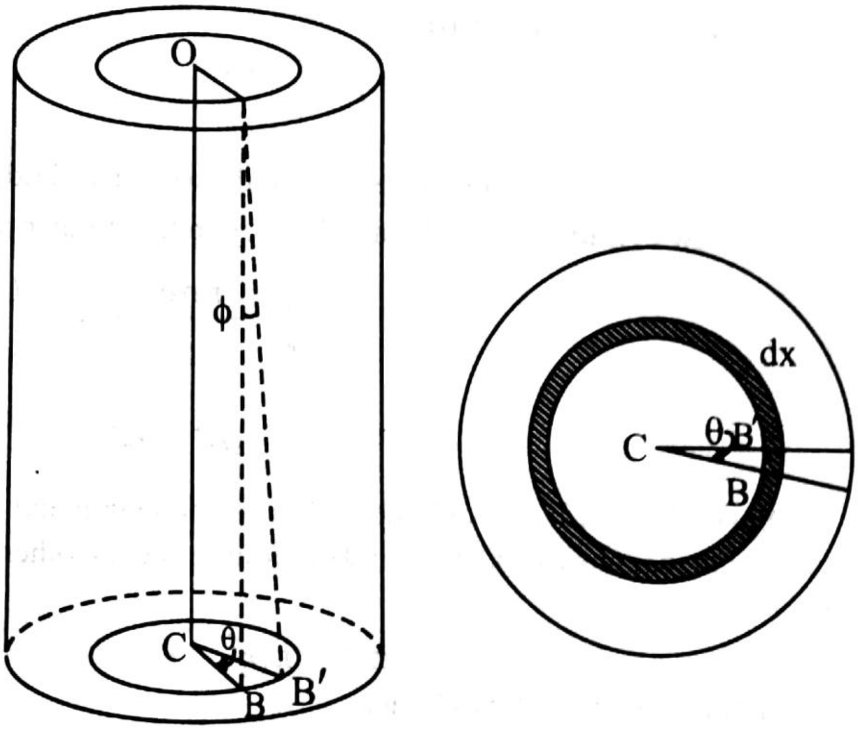
$$\frac{9}{Y}=\frac{1}{K}+\frac{3}{\eta} \quad \ldots(9)$$

which is the required relation.

6·14 Twisting of a Cylinder

Consider a cylindrical rod of length l and radius r. Its upper end is clamped and the rod is twisted by applying a couple to its lower end in a plane perpendicular to its length. The rod is said to be under torsion. A reaction is set up due to its property of elastity and a restoring couple equal and opposite to the twisting couple is produced (Fig. 6·8).

All the particles in the lower end are shifted through the same angle θ but the linear displacement of a particle near the rim is more than the particle near the centre. Hence shearing angle ϕ is more for the particles near the rim than near the axis of the cylinder.



Consider a cylinderical shell of radius x and radial thickness dx [Fig. 6·8 (*ii*)].

$$BB' = x\theta = l\phi$$

or
$$\phi = \left(\frac{x\theta}{l}\right) \qquad \ldots(i)$$

ϕ is maximum at the rim and zero at the axis.

Shearing stress $= T$

$$\eta = \frac{T}{\phi}$$

$$\because \ \eta = \frac{Tangential\ Stress}{Shear\ Strain} = \frac{F/A}{\theta} = \frac{T}{\theta}$$

$$\therefore \quad T = \eta\phi = \left(\frac{\eta x\theta}{l}\right)$$

Area of cross section of the shell $= 2\pi x dx$

Shearing force on the shell $= T \times$ area

$$F = \left(\frac{\eta x\theta}{l}\right) 2\pi x dx$$

$$F = \left(\frac{2\pi\eta\theta}{l}\right) x^2 dx$$

Moment of this force about the axis OC,

$$= \left[\left(\frac{2\pi\eta\theta}{l}\right) x^2 dx\right] x$$

$$= \left(\frac{2\pi\eta\theta}{l}\right)x^3dx$$

Total twisting couple for the whole cylinder

$$\tau = \int_0^r \left(\frac{2\pi\eta\theta}{l}\right) x^3dx = \left(\frac{\pi\eta r^4}{2l}\right)\theta \qquad \ldots(ii)$$

Let the couple per unit twist be C

$$\tau = C\theta$$

Comparing (i) and (ii)

$$C = \left(\frac{\pi\eta r^4}{2l}\right) \qquad \ldots(iii)$$

Couple per unit twist C is also called the torsional rigidity of the material of the wire.

Hollow cylinder. Consider a hollow ylinder of inner radius r_2 and outer radius r_1.

$$\therefore \qquad \tau' = \int_{r_2}^{r_1} \left(\frac{2\pi\eta\theta}{l}\right)x^2dx = \left(\frac{\pi\eta\theta}{2l}\right)(r_1^4 - r_2^4)$$

$$\therefore \qquad C' = \frac{\pi\eta}{2l}(r_1^4 - r_2^4) \qquad \ldots(iv)$$

Special Case. If two cylinders of the same length and same mass and of the same material are made such that one cylinder is solid of radius r and the other cylinder is hollow of inner radius r_2 and outer radius r_1 then,

$$\pi r^2 = \pi(r_1^2 - r_2^2)$$

In the case of a solid cylinder,

$$C = \frac{\pi\eta r^4}{2l} = \frac{\pi\eta(r_1^2 - r_2^2)^2}{2l} \qquad \ldots(i)$$

For a hollow cylinder

$$C' = \frac{\pi\eta(r_1^4 - r_2^4)}{2l} = \frac{\pi\eta(r_1^2 + r_2^2)(r_1^2 - r_2^2)}{2l} \qquad \ldots(ii)$$

Dividing (ii) by (i)

$$\frac{C'}{C} = \frac{r_1^2 + r_2^2}{r_1^2 - r_2^2} \qquad \ldots(iii)$$

$$\therefore \qquad C' > C$$

Hence, a larger twisting couple will be required in the case of a hollow cylinder as compared to a solid cylinder of the same mateal, rimass and length.

Due to this reason hollow shafts are comparatively much stronger than solid shafts. Driving shafts in automobiles are made from hollow tubes and not from solid rods.