Rotational Dynamics

It is the branch of physics that deals with the study of rotational motion along with the cause of motion.

Rigid body: A perfectly rigid body is defined as that body which does not undergo any change in shape and volume when external forces are applied on it. When external forces are applied on a rigid body the distance between any two particles of the body will remain unchanged. e.g.- solid body.

Centre of mass:

A single point where whole of the mass of the body or systems of particles is supposed to be concentrated is called centre of mass. If all the external forces are assumed to act on this point, then its motion is similar to that of the system upon which the same external forces is acting.

Conservation of linear Momentum:

The linear momentum of a particle of mass 'm' and velocity 'v' is define as

$$p = mv$$

Therefore the net linear momentum for a system of n-particles is

$$P = \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} m_i v_i$$

From Newton's 2nd we have,

$$F_{ext} = \frac{dp}{dt}$$

i.e. the rate of change of linear momentum of a system of particle is equal to the net external force acting on the system.

If F_{ext} =0 then $\frac{dp}{dt} = 0$ and integrating we gate, P = constant.

Thus if the net external force acting on a system of particles is zero, the net linear momentum of the system remains constant. This is the principle of conservation of linear momentum.

Angular Momentum: The moment of linear momentum of a rotating particle about a fixed point is called the angular momentum. It is equal to the cross product of linear momentum of the particle and the perpendicular distance from the reference point.

If a particle of mass m rotates about an axis in a circular orbit of radius r with velocity v then angular momentum of the particle is

$$\vec{L} = \vec{r} \times \vec{p} = m\left(\vec{r} \times \vec{v}\right)$$

Angular momentum is a vector quantity and is perpendicular to both $\vec{r}$ and $\vec{v}$. The SI unit of angular momentum is $Kg.m^2s^{-1}$ or $Joule.Sec$ and its dimensions are $\left[ML^2T^{-1}\right]$.

For a system of particles is $\vec{L_1}, \vec{L_2}, \vec{L_3}, \cdots\cdots\cdots$etc are the angular momentums of different particles about the reference point, the total angular momentum of the system is given by

$$\vec{L} = \vec{L_1} + \vec{L_2} + \vec{L_3} + \cdots\cdots\cdots$$

$$\Rightarrow \vec{L} = \left(\vec{r_1} \times \vec{p_1}\right) + \left(\vec{r_2} \times \vec{p_2}\right) + \left(\vec{r_3} \times \vec{p_3}\right) + \cdots\cdots\cdots$$

$$\Rightarrow \vec{L} = \sum_{n=1}^{N}\left(\vec{r_n} \times \vec{p_n}\right)$$

Therefore the angular momentum of the whole system about the reference point is the vector sum of the linear moment of all the particles of the system about that reference point.

Angular Momentum of a system of particles about centre of mass:

The total angular momentum of a system of particle can be expressed in terms of velocity of the particles with respect to centre of mass and the velocity of centre of mass.

Let $\vec{R}_{cm}$ and $\vec{V}_{cm}$ are the position vector and the velocity vector of the centre of mass of a system of particles relatives to a reference point. If $\vec{r}_{cm}$ and $\vec{v}_{cm}$ are the position vector and the velocity vector of a particle of mass 'm' relative to centre of mass then position vector $\vec{r}$ and the velocity vector $\vec{v}$ of the particle relative to reference point are

$$\vec{r} = \vec{R}_{cm} + \vec{r}_{cm} \qquad \left[\because \vec{r}_{cm} = \vec{r} - \vec{R}_{cm}\right]$$

And $\vec{v} = \vec{V}_{cm} + \vec{v}_{cm}$

Hence the total angular momentum for a system of N particles is

$$\vec{L} = \sum m\left(\vec{r}\times\vec{v}\right)$$

$$\Rightarrow \vec{L} = \sum m\left[\left(\vec{R}_{cm}+\vec{r}_{cm}\right)\times\left(\vec{V}_{cm}+\vec{v}_{cm}\right)\right]$$

$$\Rightarrow \vec{L} = \sum m\left(\vec{R}_{cm}\times\vec{V}_{cm}\right)+\sum m\left(\vec{R}_{cm}\times\vec{v}_{cm}\right)+\sum m\left(\vec{r}_{cm}\times\vec{V}_{cm}\right)+\sum m\left(\vec{r}_{cm}\times\vec{v}_{cm}\right) \quad \text{---- (1)}$$

From the property of centre of mass,

$$\sum m = M, \quad \sum m\,\vec{r}_{cm} = 0 \quad \text{and} \quad \sum m\,\vec{v}_{cm} = 0$$

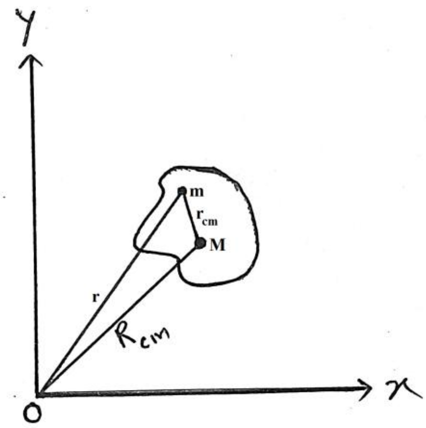
Hence from equation (1), we have

$$\therefore \vec{L} = \vec{R}_{cm}\times M\vec{V}_{cm} + 0 + 0 + \sum\left(\vec{r}_{cm}\times m\vec{v}_{cm}\right)$$

$$\Rightarrow \vec{L} = \vec{R}_{cm}\times\vec{P} + \sum\left(\vec{r}_{cm}\times\vec{p}_{cm}\right)$$

$$\Rightarrow \vec{L} = \vec{R}\times\vec{P} + \vec{L}_{cm}$$

$$\Rightarrow \vec{L}_{total} = \vec{L}_{of\ cm} + \vec{L}_{about\ cm}$$



The total angular momentum of the system is equal to sum of the angular momentum of centre of mass about the reference point and the angular momentum of the system about the centre of mass.

Torque:

The cause of linear motion is force. The corresponding quantity which is responsible for rotational motion is called torque.

Torque due to a force about the axis of rotation is defined as the moment of the force i.e. the product of magnitude of force and perpendicular distance of force from the axis of rotation.

If a particle of mass m rotates about an axis in a circular orbit of radius r with velocity v then angular momentum of the particle is

$$\vec{L} = \vec{r}\times\vec{p} = m\left(\vec{r}\times\vec{v}\right)$$

Differentiating with respect to time, we get

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

$$\Rightarrow \frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$\Rightarrow \frac{d\vec{L}}{dt} = \vec{v} \times m\vec{v} + \vec{r} \times \vec{F}$$

$$\Rightarrow \frac{d\vec{L}}{dt} = 0 + \vec{\tau}$$

The quantity $\vec{r} \times \vec{F}$ is called torque and represented by the symbol $\vec{\tau}$.

$$\therefore \quad \vec{\tau} = \vec{r} \times \vec{F} = \frac{d\vec{L}}{dt}$$

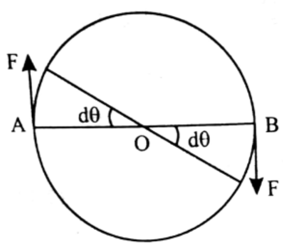
Thus, torque is equal to vector product of $\vec{r}$ & $\vec{F}$ and is equal to time rate of changed of angular momentum of a particle about a reference point.

For a system of particles,

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \sum \vec{r} \times \frac{d\vec{p}}{dt} = \sum \vec{r} \times \vec{F}$$

Work done by a Torque:

Let us consider two particles A and B on the circumference of a disc at diametrically opposite points. Let $\vec{F}$ and $-\vec{F}$ be the two forces acting on the particles A and B respectively. If the particles are displaced through $d\theta$, then displacement $dS = rd\theta$ in each case.



Therefore,

work done by each force $= FdS = Frd\theta$

Total work done $= 2Frd\theta = F(2r)d\theta = \tau\, d\theta \quad [\because F(2r) = \tau]$

If the particles get displaced through an angle θ under the action of external torque τ then total work done is

$$W = \tau\,\theta$$

When a torque τ acts for a time dt and displaces the particles through $d\theta$ the rate of work done is

$$\frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\,\omega = \text{Torque x Angular velocity.}$$

Conservation of Angular Momentum:

The moment of linear momentum of a rotating particle is called the angular momentum. The angular momentum of a particle of linear momentum $\vec{p} = m\vec{v}$ and having position vector $\vec{r}$ relative to and arbitrary origin is defined as

$$\vec{l} = \vec{r} \times \vec{p} \qquad \text{-------------------------- (1)}$$

For a system of n-particles, we have

$$\vec{L} = \sum_i \vec{l} = \sum_i \vec{r_i} \times \vec{p_i} \qquad \text{-------------------------- (2)}$$

Differentiating above equation we get,

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\sum_i (\vec{r_i} \times \vec{p_i})$$

$$\Rightarrow \frac{d\vec{L}}{dt} = \sum_i \left[\frac{d\vec{r}_i}{dt} \times \vec{p}_i + \vec{r}_i \times \frac{d\vec{p}_i}{dt} \right]$$

$$\Rightarrow \frac{d\vec{L}}{dt} = \sum_i \left[\vec{v}_i \times m_i \vec{v}_i + \vec{r}_i \times \vec{F}_i \right]$$

Where $F_i = \frac{dp_i}{dt}$ = net force acting on ith particle.

$$\therefore \frac{d\vec{L}}{dt} = \sum_i \left[0 + \vec{r}_i \times \vec{F}_i \right]$$

$$\Rightarrow \frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i \qquad \text{------------------------ (3)}$$

As internal force occurs is equal and opposite pair hence the net internal force acting on the system of particle is zero. Thus from equation (3) we have,

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i^{ext} = \vec{\tau}_{ext} \qquad \text{------------------------ (4)}$$

Where $\tau_{ext} = \sum_i r_i \times F_i^{ext}$ is the torque arising due to external force only.

If $\tau_{ext} = 0$ then $\frac{d\vec{L}}{dt} = 0$

or $\vec{L}$ = constant.

Thus in the absence of an external torque the angular momentum of a system remains constant.

Conservation of Angular Momentum under central force:

For a pair of interacting particles if the lines of action of the forces, which exert each other, lie along the straight line are called central force.

OR, Central force is the forces which act along the line of action between the pair of interacting particle. As for example, the force of Gravity, Coulomb force, the electrostatic force between the electron revolving around the nucleus and the force of spring are the central force.

Let us consider a particle subjected to a central force depending upon the distance from the fixed point. Such a force can be represented as $\vec{F} = f(r)\,\hat{r}$ where, $f(r)$ is some scalar function of the distance and $\hat{r}$ is a unit vector along $\vec{r}$ given by $\hat{r} = \frac{\vec{r}}{|r|}$.

Therefore, the torque acting on a particle under the action of central force is given by

$$\tau = \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{r} \times f(r)\hat{r} = f(r)\left(\vec{r} \times \hat{r}\right) = f(r)\left(\vec{r} \times \frac{\vec{r}}{|r|}\right) = 0 \qquad \left[\because \vec{r} \times \vec{r} = 0\right]$$

Thus, torque acting on a particle under the action of central force = 0

$$\therefore \quad \tau = \frac{d\vec{L}}{dt} = 0$$

$$\Rightarrow \vec{L} = \text{a constant.}$$

Hence the angular momentum of a particle under the influence of a central force always remains constant.

Moment of Inertia:

Inertia (in case of linear motion) is property of inability of a body to change its state of rest or of uniform motion in a straight line without the help of an external force. A corresponding property of the body, in rotational motional, is known as moment of inertia.

"Moment of inertia of a body, about a given axis is defined as the property of body by virtue of which it is unable to change its state of rest or of uniform rotational motion without the help of an external torque."

While inertia depends only upon the mass of body, moment of inertia depends upon two factors: Mass of body and Distribution of mass about the axis of rotation. Moment of inertia (MI) plays the same role in rotational motion as is played by mass in linear motion. It is a scalar quantity.

Let us consider a body of mass M rotating about the axis YY' with an angular velocity ω (Fig:1). Let $m_1, m_2, m_3, \cdots\cdots\cdots$ be the masses of various particles situated at distances $r_1, r_2, r_3, \cdots\cdots\cdots$ respectively from the axis of rotation. If $v_1, v_2, v_3, \cdots\cdots\cdots$ respectively be their linear velocity then

$$v_1 = r_1\omega, v_2 = r_2\omega, v_3 = r_3\omega, \cdots\cdots\cdots$$

Therefore, total kinetic energy of the body is given by

$$E = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + \cdots\cdots\cdots\cdots$$

$$\Rightarrow E = \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + \frac{1}{2}m_3r_3^2\omega^2 + \cdots\cdots\cdots\cdots$$

$$\Rightarrow E = \frac{1}{2}\omega^2\left(m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \cdots\cdots\cdots\cdots\right)$$

$$\Rightarrow E = \frac{1}{2}\omega^2\sum mr^2$$

$$\Rightarrow E = \frac{1}{2}I\omega^2$$ (This is the expression of Kinetic energy of rotation)

Here, $I = \sum mr^2$ is called the moment of inertia of the body.

$$\therefore\ I = \frac{2E}{\omega^2}$$

If $\omega = 1$ then $I = 2E$ i.e. moment of inertia of a body is equal of twice the kinetic energy of rotation of a body whose angular velocity is 1 radian/s.

Suppose the body consist of n particles each of mass 'm'.

$$I = \sum mr^2 = m\left(r_1^2 + r_2^2 + r_3^2 + \cdots\cdots\cdots + r_n^2\right)$$

$$\Rightarrow I = m \times n\frac{\left(r_1^2 + r_2^2 + r_3^2 + \cdots\cdots\cdots + r_n^2\right)}{n}$$

$$\Rightarrow I = MK^2$$

Where, $K^2 = \frac{\left(r_1^2 + r_2^2 + r_3^2 + \cdots\cdots\cdots + r_n^2\right)}{n}$

$$\Rightarrow K = \sqrt{\frac{\left(r_1^2 + r_2^2 + r_3^2 + \cdots\cdots\cdots + r_n^2\right)}{n}}$$

Here K is called the radius of gyration and is equal to the root mean square distance of the particles from the axis of rotation.

Relation between Torque and Moment of Inertia:

Let us consider a particle of mass 'm' moving about an axis in a circular path of radius 'r'. let an external force F act on the particle along the tangent to the circular path. Now the moment of force i.e. torque acting on the particle is

$$\vec{\tau} = \vec{r} \times \vec{F} = rF\sin 90^0$$

$$\Rightarrow \vec{\tau} = rF = rm\vec{a}$$

$$\Rightarrow \vec{\tau} = rmr\vec{\alpha}$$

$$\Rightarrow \vec{\tau} = mr^2\vec{\alpha}$$

$$\Rightarrow \vec{\tau} = I\vec{\alpha}$$

Theorem of Perpendicular Axes

(*a*) **For a Plane Lamina. The theorem of** perpendicular axes states that the moment of inertia of a plane laminar body about an axis perpendicular to the plane is equal to the sum of the moments of inertia about two mutually perpendicular axes inthe plane of the lamina such that the three mutually perpendicular axes have a common **point of intersection**.

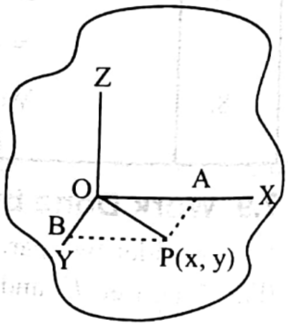


Fig. 7.5

Consider a plane lamina having the axes *OX* and *OY* in the plane of the lamina. The axis *OZ* passes through *O* and is perpendicular to the plane of the lamina. Let the lamina be divided into a large number of particles, each of mass *m*. Let a particle of mass *m* be at *P* with coordinates (*x*, *y*) and situated at a distance *r* from the point of intersection of the axes (Fig. 7.5).

$$\therefore \quad r^2 = x^2 + y^2 \qquad \text{...(7.13)}$$

The moment of inertia of the particle *P* about the axis $OZ = mr^2$

The moment of inertia of the whole lamina about the axis *OZ* is given by

$$I_z = \Sigma mr^2 \qquad \text{...(7.14)}$$

The moment of inertia of the whole lamina about the axis *OX*,

$$I_x = \Sigma my^2 \qquad \text{...(7.15)}$$

Similarly,

$$I_y = \Sigma mx^2 \qquad \text{...(7.16)}$$

From equation (7.14)

$$I_z = \Sigma mr^2 = \Sigma m(x^2 + y^2) \qquad \because r^2 = x^2 + y^2$$

$$I_z = \Sigma mx^2 + \Sigma my^2 = I_y + I_x$$

$$\therefore \quad I_z = I_x + I_y \qquad \text{...(7.17)}$$

(*b*) **For a three-dimensional body.** Consider a threedimensional body. The three mutually perpendicular axes meet at the point *O* inside the body. A partricle *P* (*x*, *y*, *z*) of mass *m* is at a distance *r* from *O* (Fig. 7.6).

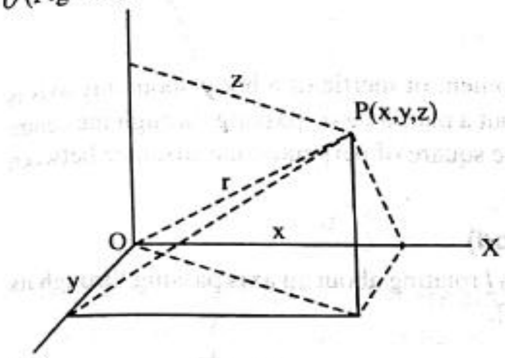


Fig. 7.6

$$\therefore \quad r^2 = x^2 + y^2 + z^2 \qquad \text{...(7.18)}$$

The moment of inertia of the body about the *Z*-axis,

$$I_z = \Sigma m(x^2 + y^2) \qquad \text{...(7.19)}$$

$$I_x = \Sigma m(y^2 + z^2) \qquad \text{...(7.20)}$$

$$I_y = \Sigma m(x^2 + z^2) \qquad \text{...(7.21)}$$

The moment of inertia of the body about any axis passing through *O*,

$$I = \Sigma mr^2 = \Sigma m(x^2 + y^2 + z^2) \qquad \text{...(7.22)}$$

Adding (7.18), (7.19), (7.20) and (7.21)

$$I_x + I_y + I_z = \Sigma m(x^2 + y^2) + m(y^2 + z^2) + m(x^2 + z^2)$$

$$= 2\Sigma m(x^2 + y^2 + z^2) = 2\Sigma mr^2 = 2I$$

or

$$I = \frac{1}{2}(I_x + I_y + I_z)$$

Thus, the moment of Inertia of a three dimensional body about any axis is equal to half the sum of the moments of inertia about three mutually perpendicular axes about the common point of intersection.

7.11 Theorem of Parallel Axes

Consider a plane laminar body having its centre of gravity at G. The axis XX' passes through the centre of gravity and is perpendicular to the plane of the lamina. The axis X_1X_1' passes through the point O and is parallel to the axis XX'. The distance between the two parallel axes is x (Fig. 7.7).

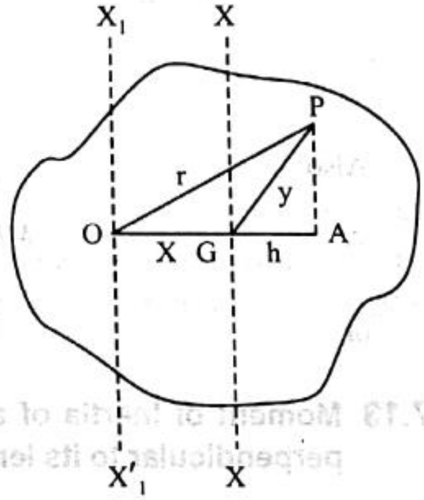


Fig. 7.7

Let the lamina be divided into large number of particles each of mass m. The moment of inertia on the particle of mass m at P about the axis X_1X_1' is equal to mr^2.

The moment of inertia of the whole lamina about the axis X_1X_1' is given by

$$I_0 = \Sigma mr^2$$

In the ΔOPA

$$OP^2 = (OA)^2 + (AP)^2$$

$$r^2 = (x + h)^2 + (AP)^2$$

$$r^2 = x^2 + 2xh + h^2 + (AP)^2$$

$$r^2 = x^2 + y^2 + 2xh$$

$$I_0 = \Sigma m(x^2 + y^2 + 2xh)$$

$$I_0 = \Sigma mx^2 + \Sigma my^2 + \Sigma m(2xh)$$

$$I_0 = Mx^2 + I_G + 2x\Sigma mh$$

Here $\qquad \Sigma my^2 = I_G$

and $\qquad \Sigma mh = 0,$

because the body balances about centre of mass at G. Therefore, the algebraic some of moments of all the particles about the centre of gravity *i.e.*,

$$\Sigma mgh = 0,$$

As g is constant

$$\Sigma mh = 0$$

Hence $\qquad \vec{I_0} = \vec{I}g + \vec{M}\vec{x}^2$

Thus, the theorem of parallel axes states that the moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through the centre of gravity and the product of the mass of the body and the square of perpendicular distance between the two parallel axes.

7.12 Moment of Inertia of a Thin Uniform Bar (Rod)

Consider a thin uniform bar AB of mass M and length l rotating about an axis passing through its centre and perpendicular to its length (axis YY') [Fig. 7.8].

Mass of the bar $= M$

Length of bar $= l$

Mass per unit length $= \left(\frac{M}{l}\right)$

Take an element of length dx at a distance x from the axis.

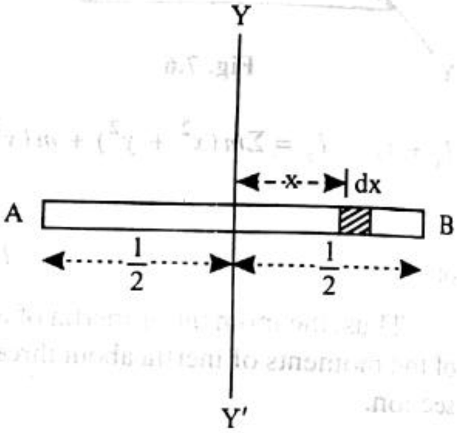


Fig. 7.8

Mass of the element $= \left(\frac{M}{l}\right) dx$

Moment of inertia of the element about the axis YY'

$$= \left[\left(\frac{M}{l}\right)dx\right] x^2$$

Moment of inertia of the bar AB about the axis YY'

$$I = 2\int_0^{l/2} \left(\frac{M}{l}\right) x^2\, dx = \frac{2M}{l}\left[\frac{x^3}{3}\right]_0^{\frac{l}{2}} \quad \text{[Factor of 2 ∵ taken on both sides]}$$

$$I = \frac{Ml^2}{12} \quad \text{...(7.23)}$$

Also $\quad I = MK^2$

∴ $\quad MK^2 = \frac{Ml^2}{12}$

or $\quad K = \frac{l}{2\sqrt{3}} \quad$ ...(7.24)

7.13 Moment of Inertia of a Bar *AB* about an Axis passing through one end and perpendicular to its length (Fig. 7.9).

Here $\quad I = \int_0^l \left(\frac{M}{l}\right) x^2\, dx$

$$I = \frac{Ml^2}{3} \quad \text{...(7.25)}$$

But $\quad I = MK^2$

∴ $\quad MK^2 = \frac{Ml^2}{3}$

$$K = \frac{l}{\sqrt{3}} \quad \text{...(7.26)}$$

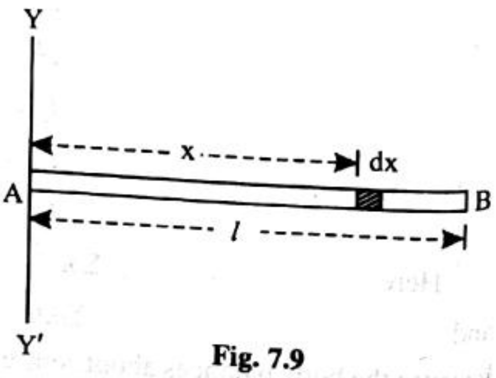


Fig. 7.9

7.14 Moment of Inertia of a Bar About an Axis Perpendicular to its Length, at a Distance *a* from One End (Fig. 7.10)

Here $\quad I = \int_{-a}^{(l-a)} \left(\frac{M}{l}\right) x^2\, dx$

$$I = \frac{M}{l}\left[\frac{x^3}{3}\right]_{-a}^{l-a}$$

$$I = M\left[\frac{2}{3} - la + a^2\right] \quad \text{...(7.27)}$$

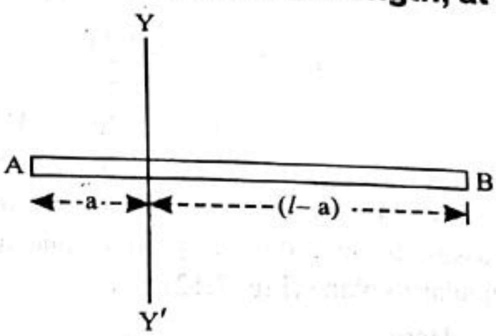


Fig. 7.10

7.15 Moment of Inertia of a Rectangular Lamina (i.e. a thin plate or scale or thin plane sheet)

(*a*) Consider a rectangular lamina of uniform thickness, length l and breadth b. Consider an element of length dx at a distance x from the axis YY' (Fig. 7.11).

Mass of the lamina $= M$

Area of the lamina $= l \times b$

Mass per unit area $= \left(\frac{M}{l \times b}\right)$

Area of the element $= b \times dx$

Mass of the element $= \left(\frac{M}{l \times b}\right) b \times dx = \left(\frac{M}{l}\right) dx$

Moment of Inertia of the element about the axis YY'

$$= \left(\frac{M}{l} dx\right) x^2$$

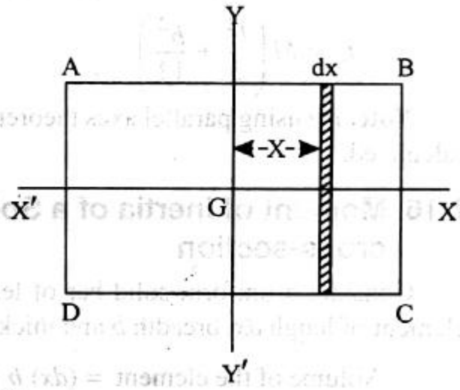


Fig. 7.11

Moment of inertia of the whole lamina about the axis YY',

$$I_y = \int_{-l/2}^{+l/2} \frac{M}{l} x^2 dx = \frac{Ml^2}{12} \quad ...(7.28)$$

Similarly, moment of inertia of the lamina about the axis XX',

$$I_x = \frac{Mb^2}{12} \quad ....(7.29)$$

(*b*) The moment of inertia of the lamina about an axis (ZZ') passing through the centre of gravity and perpendicular to the plane of the lamina, can be calculated by applying perpendicular axes theorem.

$$\therefore \quad I_z = I_x + I_y$$

$$I_z = \frac{M}{12}(l^2 + b^2) \quad ...(7.30)$$

$$I_z = MK^2$$

$$MK^2 = \frac{M}{12}(l^2 + b^2)$$

$$K = \sqrt{\frac{(l^2 + b^2)}{12}} \quad ...(7.31)$$

(*c*) Moment of inertia of the lamina about an axis AD

$$I_{AD} = I_y + M\left(\frac{l}{2}\right)^2$$

$$I_{AD} = \frac{Ml^2}{12} + \frac{Ml^2}{4} = \frac{Ml^2}{3}$$

(*d*) Moment of inertia of the lamina about an axis AB

$$I_{AB} = I_x + M\left(\frac{b}{2}\right)^2$$

$$= \frac{Mb^2}{12} + \frac{Mb^2}{4} = \frac{Mb^2}{3}$$

(*e*) Moment of inertia of the lamina about an axis passing through the mid-point of one side and perpendicular to plane (Fig. 7.12).

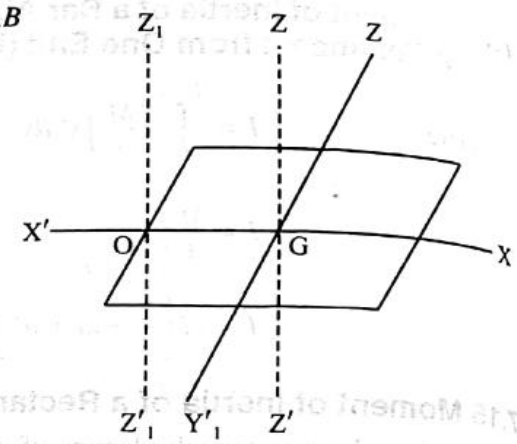


Fig. 7.12

Here

$$I_0 = I_z + M\left(\frac{l}{2}\right)^2$$

$$I_0 = \frac{M(l^2 + b^2)}{12} + \frac{Ml^2}{4}$$

$$I_0 = M\left(\frac{l^2}{3} + \frac{b^2}{12}\right)$$

Note. By using parallel axes theorem, the moment of inertia of the lamina about any axis can be calculated.

7.16 Moment of Inertia of a Solid Uniform Bar of Rectangular cross-section

Consider a uniform solid bar of length, l, breadth b and thickness t (Fig. 7.13). Consider an element of lengh dx, breadth b and thickness t from the axis YY'.

Volume of the element $= (dx)\, b \times t$

Mass of the element $= \left(\dfrac{M}{lbt}\right) dx\, bt = \left(\dfrac{Mdx}{l}\right)$

$$I_y = \int_{-l/2}^{l/2} \frac{M}{l} x^2 dx = \frac{Ml^2}{12}$$

Similarly $\quad I_x = \dfrac{Mb^2}{12}$

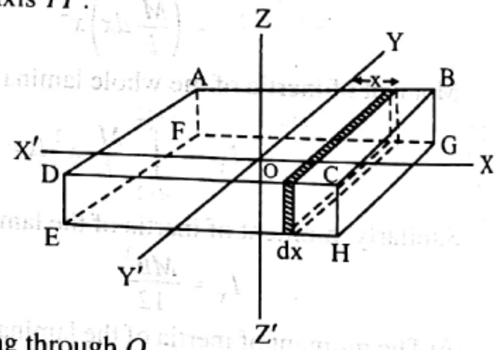


Fig. 7.13

∴ Moment of inertia of the bar about an axis passing through O,

$$I_z = I_x + I_y = \frac{Mb^2}{12} + \frac{Ml^2}{12} = \left(\frac{l^2 + b^2}{12}\right)$$

Similarly the moment of inertia of the bar about an axis perpendicular to other faces can be calculated. In that case it will be,

$$M\left(\frac{l^2 + t^2}{12}\right) \text{ and } M\left(\frac{b^2 + t^2}{12}\right)$$

7.20 Moment of Inertia of a Hollow Cyliner

(a) About its own axis

Consider a cylinder of length l and mass M. Its inner radius is R_1 and outer radius is R_2 (Fig. 7.17).

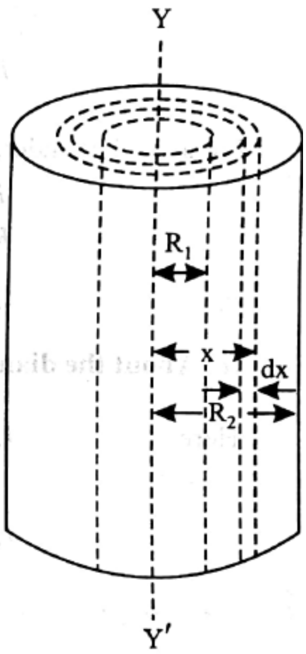


Fig. 7.17

Mass per unit volume of the the cylinder

$$= \frac{M}{\pi(R_2^2 - R_1^2)l}$$

Consider an element of length l, radius x and radial thickness dx.

Volume of the element

$$= (2\pi x) \times dx \times l$$

Mass of the element

$$= \frac{M}{\pi} \frac{(2\pi x\, dx) l}{(R_2^2 - R_1^2) l}$$

$$= \frac{2 Mx\, dx}{(R_2^2 - R_1^2)}$$

Moment of inertia of the element about the axis YY'

$$= \left(\frac{2 Mx\, dx}{R_2^2 - R_1^2}\right) x^2$$

Moment of inertia of the whole cylinder about the axis YY',

$$I = \int_{R_1}^{R_2} \left(\frac{2 Mxdx}{R_2^2 - R_1^2}\right) x^2$$

$$I = \frac{M\,(R_2^2 + R_1^2)}{2} \qquad \text{...7.38}$$

Special case. For a solid cylinder,

$$R_1 = 0$$

and

$$R_2 = R$$

∴

$$I = \frac{MR^2}{2}$$

(b) About an axis passing through its centre and perpendicular to its length.

(i) *Hollow cylinder.* Let AB be the axis of symmetry of a hollow cylinder of mass M length l and internal and external radii R_1 and R_2 respectively [Fig. 6.19]. XY is the axis passing through the centre and perpendicular to the length of the cylinder.

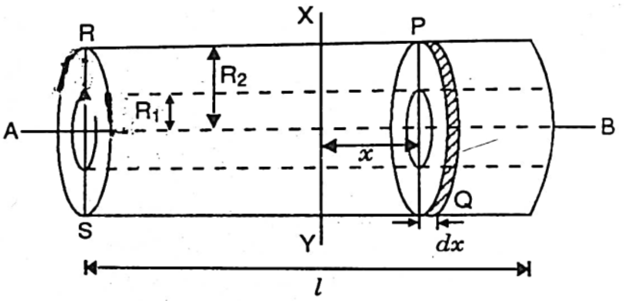


Fig. 6.19 Moment of inertia of a cylinder about an axis passing through its centre and perpendicular to its length.

Volume of cylinder $= \pi(R_2^{\,2} - R_1^{\,2})l$

Mass per unit volume $= \dfrac{M}{(R_2^{\,2} - R_1^{\,2})l}$

Consider an element of the cylinder in the form of an annular disc situated at a distance x from XY and having a thickness dx. Axis of symmetry passes through the centre of the disc and is perpendicuiar to the plane of disc.

Volume of element $= \pi(R^2 - R_1^{\,2})\, dx$

$\therefore$ Mass of the element $= \dfrac{M}{\pi(R_2^{\,2} - R_1^{\,2})\, l} \times \pi(R_2^{\,2} - R_1^{\,2})\, dx = \dfrac{M}{l}\, dx.$

If dI_g is the moment of inertia of the element (annular disc) about its own diameter PQ (parallel to XY),

$$dI_g = \frac{1}{4}\left(\text{mass of the disc}\right)\left(R_2^{\,2} + R_1^{\,2}\right)$$

$\therefore$
$$dI_g = \frac{1}{4}\left[\frac{M}{l}\, dx\right][R_2^{\,2} + R_1^{\,2}]$$

Moment of inertia 'dI' of the element about XY, can be calculated by using parallel axes theorem.

$$dI = dI_g + \left(\frac{M}{l}\, dx\right)x^2$$

or
$$dI = \frac{1}{4}\left[\frac{M}{l}\, dx\right][R_2^{\,2} + R_1^{\,2}] + \frac{M}{l}\, dx\, x^2$$

or
$$dI = \frac{M}{l}\left[\frac{R_2^{\,2} + R_1^{\,2}}{4} + x^2\right]dx$$

Moment of inertia I of the cylinder can be obtained by integrating the above expression between the limits $x = -\dfrac{l}{2}$ to $x = +\dfrac{l}{2}$

$\therefore$
$$I = \int_0^I dI = \frac{M}{l}\left(\frac{R_2^{\,2} + R_1^{\,2}}{4}\right)\int_{-l/2}^{+l/2} dx + \frac{M}{l}\int_{-l/2}^{+l/2} x^2\, dx$$

$$= \frac{M}{l}\left[\frac{R_2^{\,2} + R_1^{\,2}}{4}\right]\left[x\right]_{-l/2}^{+l/2} + \frac{M}{l}\left[\frac{x^3}{3}\right]_{-l/2}^{+l/2}$$

$$= \frac{M}{l}\left[\frac{R_2^{\,2} + R_1^{\,2}}{4}\right]\left[\left(\frac{l}{2}\right) - \left(-\frac{l}{2}\right)\right] + \frac{M}{3l}\left[\left(\frac{l}{2}\right)^3 - \left(-\frac{l}{2}\right)^3\right]$$

$$= \frac{M}{l}\left[\frac{R_2^{\,2} + R_1^{\,2}}{4}\right] l + \frac{M}{3l}\left[2 \times \frac{l^3}{8}\right] = M\left(\frac{R_2^{\,2} + R_1^{\,2}}{4}\right) + \frac{Ml^2}{12}$$

or
$$I = M\left[\frac{R_2^{\,2} + R_1^{\,2}}{4} + \frac{l^2}{12}\right] \qquad \text{...(21)}$$

(*ii*) *Solid cylinder.* In case of a solid cylinder,

$$R_1 = 0, R_2 = R$$

Making the substitutions in equation (21),

$$1 = M\left[\frac{R^2}{4} + \frac{l^2}{12}\right] \qquad \text{...(22)}$$

(*c*) **About a diameter of its face**

(*i*) *Hollow cylinder.* As shown in Fig. [1(*b*).19], *RS* is the axis co-inciding with a diameter of one of its face.

Using parallel axes theorem,

$$I_{RS} = I_{XY} + Mh^2$$

$$I_{XY} = M\left[\left(\frac{R_2{}^2 + R_1{}^2}{4}\right) + \frac{l^2}{12}\right] \qquad [=n\ (21)]$$

and

$$h = \frac{l}{2}$$

$$\therefore \qquad I_{RS} = M\left[\left(\frac{R_2{}^2 + R_1{}^2}{4}\right) + \frac{l^2}{12}\right] + M\left(\frac{l}{2}\right)^2$$

$$= M\left[\frac{R_2{}^2 + R_1{}^2}{4} + \frac{l^2}{12} + \frac{l^2}{4}\right]$$

or

$$I_{RS} = M\left[\frac{R_2{}^2 + R_1{}^2}{4} + \frac{l^2}{3}\right] \qquad \text{...(23)}$$

(*ii*) *Solid cylinder.* In case of a solid cylinder,

$$R_1 = 0, R_2 = R$$

$$I_{RS} = M\left[\frac{R^2}{4} + \frac{l^2}{3}\right] \qquad \text{...(24)}$$

(b) About an axis passing through the centre and perpendicular to the length of the cylinder (Fig. 7.18).

Moment of the inertia of the element about the axis AB,

$$dI_A = \left(\frac{M}{l}\,dx\right)\left(\frac{R_2^2 + R_1^2}{4}\right)$$

Moment of inertia of the element about the axis XX'

$$dI_x = dI_A + \left(\frac{M}{l}\,dx\right)x^2$$

$$dI_x = \left(\frac{M}{l}\right)\left[\frac{R_2^2 + R_1^2}{4} + x^2\right]dx$$

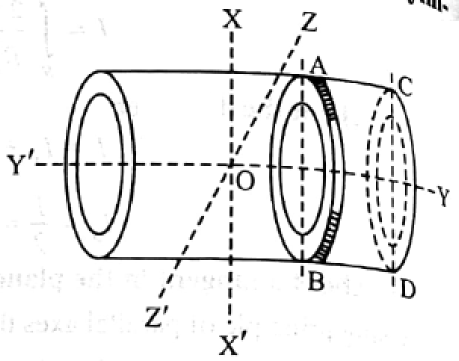


Fig. 7.18

Moment of inertia of the whole cylinder about the axis XX'

$$I_x = \int_{-l/2}^{+l/2} \frac{M}{l}\left[\left(\frac{R_2^2 + R_1^2}{4}\right) + x^2\right]dx$$

$$I_x = M\left[\left(\frac{R_2^2 + R_1^2}{4}\right) + \frac{l^2}{12}\right]$$

For a solid cylinder,

$$R_1 = 0$$

$$R_2 = R$$

$$\therefore \quad I_x = M\left[\frac{R^2}{4} + \frac{l^2}{12}\right] \qquad \text{...(7.39)}$$

(c) About the diameter of its one face.

Here $\quad I_{CD} = I_x + M\left(\frac{l}{2}\right)^2$

$$I_{CD} = M\left[\frac{(R_2^2 + R_1^2)}{4} + \frac{l^2}{12}\right] + \frac{Ml^2}{4}$$

$$= M\left[\frac{(R_2^2 + R_1^2)}{4} + \frac{l^2}{3}\right] \qquad \text{...(7.40)}$$

For a solid cylinder,

$$R_1 = 0,$$

$$R_2 = R$$

$$\therefore \quad I_{CD} = M\left[\frac{R^2}{4} + \frac{l^2}{3}\right] \qquad \text{...(7.41)}$$

7.21 Moment of Inertia of a Solid sphere

(i) About the diameter.

Consider a solid sphere of radius R and mass M (Fig. 7.19).

Volume of the sphere $= \frac{4}{3}\pi R^3$

Mass per unit volume $= \dfrac{M}{\frac{4}{3}\pi R^3}$

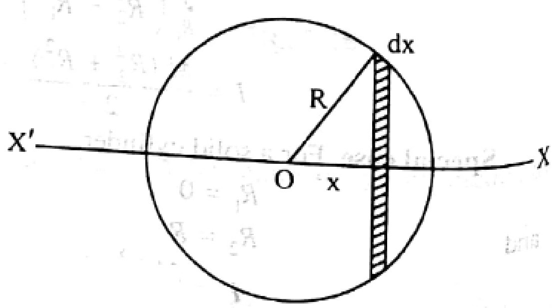


Fig. 7.19

Consider an element of thickness dx at a distance x from the centre.

Radius of the element $= r = \sqrt{R^2 - x^2}$

Volume of the element $= \pi r^2 dx$

$$\text{Mass of the element} = \frac{M \times \pi r^2\, dx}{\frac{4}{3}\pi R^3} = \frac{3M\,(R^2 - x^2)\,dx}{4R^3}$$

Moment of inertia of the element about the axis XX'

$$= \left[\frac{3M\,(R^2 - x^2)}{4R^3}\right]\frac{r^2}{2}dx = \frac{3M\,(R^2 - x^2)\,(R^2 - x^2)\,dx}{8R^3}$$

$$= \frac{3M}{8R^3}\,[R^2 - x^2]^2\,dx$$

Moment of inertia of the whole sphere about the axis XX',

$$I = 2\int_0^R \frac{3M}{8R^3}\,(R^2 - x^2)\,dx = \frac{3M}{4R^3}\int_0^R R^4 dx\,.\,x^4 dx - 2R^2x^2 dx$$

$$I = \frac{3M}{4R^3}\left(R^5 + \frac{R^5}{5} - \frac{2R^5}{3}\right) = \frac{3M}{4R^3} \times \frac{8R^5}{15} = \frac{2}{5}MR^2$$

$$\therefore \quad I = \frac{2}{5}MR^2 \qquad \text{...(7.42)}$$

Also $\quad I = MK^2$

$$\therefore \quad MK^2 = \frac{2}{5}MR^2$$

$$K = (\sqrt{0.4})\,R$$

In general, as the diameter is symmetrical, the moment of intertia of a solid sphere about any diameter $= \frac{2}{5}MR^2$

(*b*) **About a tangent** (Fig. 7.20).

Moment of inertia of the solid sphere about the tangent YY' is given by

$$I_y = I_{AB} + MR^2$$

$$I_y = \frac{2}{5}MR^2 + MR^2$$

$$I_y = \frac{7}{5}MR^2 \qquad \text{...(7.43)}$$

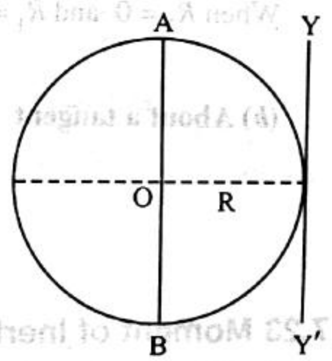


Fig. 7.20

7.22 Moment of Inertia of a Hollow Sphere

(a) About the diameter

Consider a hollow sphere of inner radius R_2 and outer radius R_1 (Fig. 7.21). Mass of the sphere = M. Let the density of the material be ρ.

$$\therefore \quad M = \frac{4}{3}\pi(R_1^3 - R_2^3)\rho \quad ...(7.44)$$

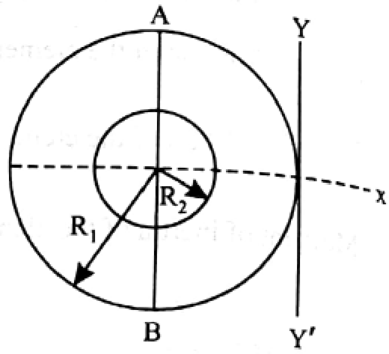


Fig. 7.21

Moment of inertia of the hollow sphere is equal to the M.I. of the solid sphere of radius R_1 minus the M.I. of the removed solid sphere of radius R_2.

Therefore, moment of inertia of the hollow sphere,

$$I = \frac{2}{5}(M_1R_1^2 - M_2R_2^2)$$

$$I = \frac{2}{5}\left[\frac{4}{3}\pi R_1^3\rho R_1^2 - \frac{4}{3}\pi R_2^3\rho R_2^2\right]$$

$$\therefore \quad I = \frac{2}{5}\times\frac{4}{5}\pi\rho(R_1^5 - R_2^5) \quad ...(7.45)$$

From equation 7.44

$$\rho = \frac{M}{\frac{4}{3}\pi(R_1^3 - R_2^3)}$$

Substituting the value of ρ in equation 7.45, we get

$$I = \frac{\frac{2}{5}M(R_1^5 - R_2^5)}{(R_1^3 - R_2^3)} \quad ...(7.46)$$

Special case

When $R_2 = 0$ and $R_1 = R$

$$I = \frac{2}{5}MR^2 \quad ...(7.47)$$

(b) About a tangent

$$I_y = I_{AB} + MR_1^2$$

$$= \frac{2}{5}\left(\frac{R_1^5 - R_2^5}{R_1^3 - R_2^3}\right) + MR_1^2 \quad ...(7.48)$$

7.23 Moment of Inertia of a Spherical Shell

(a) About a diameter

Consider spherical shell of radius R and mass M. Consider an element between two planes at P and Q (Fig. 7.22). The distance between the two planes is dx.

The radius of the element

$$y = R\cos\theta$$

Also

$$x = R\sin\theta \quad \therefore dx = R\cos\theta\, d\theta$$

$$EG = R.d\theta$$

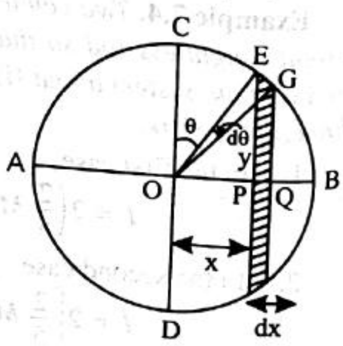


Fig. 7.22

Surface area of the element $= 2\pi y\,(EG)$

$$= 2\pi R\cos\theta . Rd\theta$$

$$= 2\pi R^2\cos\theta\, d\theta = 2\pi R dx$$

Mass per unit area of the shell $= \dfrac{M}{4\pi R^2}$

Mass of the element $= \frac{M}{4\pi R^2} \times 2\pi\, R dx = \frac{M\, dx}{2R}$

Moment of Inertia of the element about the diameter AB

$$= \left(\frac{M\, dx}{2R}\right) y^2 = \frac{M\, dx}{2R}(R^2 - x^2)$$

M.I. of the whole shell about the diameter AB,

$$I = 2\int_0^R \frac{M}{2R}(R^2 - x^2)\, dx$$

$$I = \frac{M}{R}\left(R^3 - \frac{R^3}{3}\right) = \frac{2}{3} MR^2 \qquad \text{...(7.49)}$$

(b) About the tangent

$$I_t = I + MR^2$$

$$I_t = \frac{2}{3}MR^2 + MR^2 = \frac{5}{3}MR^2 \qquad \text{...(7.50)}$$

Motion involving both translation and rotation:

Translational motion: When force acts on a stationary rigid body which is free to move, the body starts moving in a straight path in the direction of force is called translational motion.

Rotational motion: If a body is fixed (pivoted/Hinge) at a point and the force is applied on the body at suitable point, it rotates the body about the axis of rotation is called rotational motion.

****