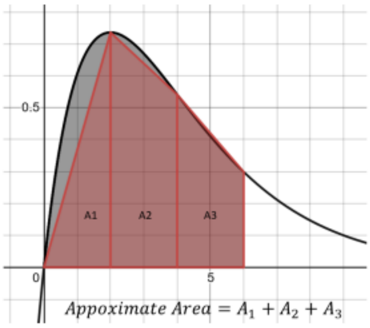
Numerical Methods: The Trapezoidal Rule



You will have evaluated definite integrals such as

$$\int_1^3 \left(x^2\right) dx$$

before. In doing this, you are evaluating the area under the graph of $f(x) = x^2$ between $x = 1$ and $x = 3$. This is only possible if you can find an antiderivative for x^2. In this example it is easy, the antiderivative is

$$F(x) = \frac{1}{3}x^3 + c, \text{ where } c \text{ is a constant.}$$

and

$$\begin{aligned}\int_1^3 \left(x^2\right) dx &= F(3) - F(1) \\ &= \frac{1}{3}3^3 + c - \left(\frac{1}{3}1^3 + c\right) \\ &= \frac{26}{3}.\end{aligned}$$

Sometimes it is not possible to find the antiderivative. In such cases you need to use a numerical method. Two common methods for calculating definite integrals are:

1. Simpson's rule, and
2. The trapezoidal rule.

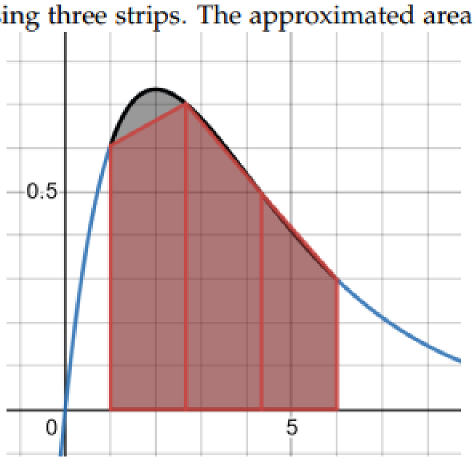
This module considers the trapezoidal rule.

The Trapezoidal Rule

The trapezoidal rule works by estimating the area under a graph by a series of trapezoidal strips. In the figure below, we see an approximation to

$$\int_1^6 xe^{-0.5x} dx$$

using three strips. The approximated area is shown in red.



In this case, we see the trapezoidal rule will underestimate the first strip, is close in the second strip and will overestimate in the third strip. The trapezoidal rule approximates the area under the curve by adding the areas of the trapezoids. Any number of strips may be used. The accuracy increases as the number of strips increases.

For the definite integral

$$\int_a^b f(x)\,dx$$

the trapezoidal rule has the form

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n\right] \qquad (1)$$

where

1. n is the number of strips and can be any number.
2. $y_n = f(x_n)$ are the values of $f(x_n)$ at the points x_i where $i = 0, 1, 2, \ldots, n$. Note that $x_0 = a$, $x_n = b$.
3. h is the width of each strip and
$$h = \frac{b-a}{n}.$$
4. $x_1 = a + h$, $x_2 = a + 2h$, $x_3 = a + 3h, \ldots$ and so on.

Example 1

Approximate $\int_0^6 xe^{-0.5x}dx$ using the trapezoidal rule with 3 strips , to 3 decimal places.

We have $n = 3$, $a = x_0 = 0$ and $b = x_3 = 6$ so

$$\begin{aligned} h &= \frac{b-a}{n} \\ &= \frac{6}{3} \\ &= 2. \end{aligned}$$

$x_0 = 0$ and $y_0 = f(x_0) = 0$.
$x_1 = 0 + h = 2$ and $y_1 = f(x_1) = 2e^{-1} = 0.73576$.
$x_2 = 0 + 2h = 4$ and $y_2 = f(x_2) = 4e^{-2} = 0.54134$.
$x_3 = 0 + 3h = 6$ and $y_3 = f(x_3) = 6e^{-3} = 0.29872$.
Substituting into (1) above we get:

$$\int_0^6 xe^{-0.5x}dx \approx \frac{2}{2}\left[y_0 + 2y_1 + 2y_2 + y_3\right]$$
$$= 0 + 2\,(0.73576) + 2\,(0.54134) + 0.29872$$
$$= 2.85292.$$

Hence, to three decimal places, $\int_0^6 xe^{-0.5x}dx \approx 2.853$.

Example 2

Approximate $\int_0^1 \sqrt{1+x^3}dx$ using the trapezoidal rule with 5 strips , to 3 decimal places.

In this question, $n = 5$, $a = 0$ and $b = 1$ so

$$h = \frac{b-a}{n} = \frac{1}{5} = 0.2\,.$$

So using the formula above we get

$x_0 = 0$ and $y_0 = f\,(x_0) = 1$.

$x_1 = 0 + h = 0.2$ and $y_1 = f\,(x_1) = \sqrt{1+(0.2)^3} = 1.00399$.

$x_2 = 0 + 2h = 0.4$ and $y_2 = f\,(x_2) = \sqrt{1+(0.4)^3} = 1.03150$.

$x_3 = 0 + 3h = 0.6$ and $y_3 = f\,(x_3) = \sqrt{1+(0.6)^3} = 1.10272$.

$x_4 = 0 + 4h = 0.8$ and $y_4 = f\,(x_4) = \sqrt{1+(0.8)^3} = 1.22963$.

$x_5 = 0 + 5h = 1.0$ and $y_5 = f\,(x_5) = \sqrt{1+(1)^3} = 1.41421$.

Substituting into Substituting into (1) above we get:

$$\int_0^1 \sqrt{1+x^3}dx \approx \frac{0.2}{2}\left[y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + y_5\right]$$
$$= 0.1\left[1 + 2\,(1.00399) + 2\,(1.03150)\right.$$
$$\left.+2\,(1.10272) + 2\,(1.22963) + 1.41421\right]$$
$$= 1.11499.$$

Hence, to three decimal places, $\int_0^1 \sqrt{1+x^3}dx \approx 1.115$.

Exercises

1. Use the trapezoidal rule to evaluate the integral $\int_1^3 (3x^2 + 4x)\,dx$ using 3 strips to two decimal places.
2. Use the trapezoidal rule to evaluate the integral $\int_0^1 (3e^x \sin(x))\,dx$ using 2 strips to two decimal places.

Answers

1. 42.44
2. 2.90

Simpson's 1/3 Rule of Integration

What is integration?

Integration is the process of measuring the area under a function plotted on a graph. Why would we want to integrate a function? Among the most common examples are finding the velocity of a body from an acceleration function, and displacement of a body from a velocity function. Throughout many engineering fields, there are (what sometimes seems like) countless applications for integral calculus. You can read about some of these applications in Chapters 07.00A-07.00G.

Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods has been developed to simplify the integral. Here, we will discuss Simpson's 1/3 rule of integral approximation, which improves upon the accuracy of the trapezoidal rule.

Here, we will discuss the Simpson's 1/3 rule of approximating integrals of the form

$$I = \int_a^b f(x)dx$$

where

$f(x)$ is called the integrand,

$a =$ lower limit of integration

$b =$ upper limit of integration

Simpson's 1/3 Rule

The trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial over interval of integration. Simpson's 1/3 rule is an

extension of Trapezoidal rule where the integrand is approximated by a second order polynomial.

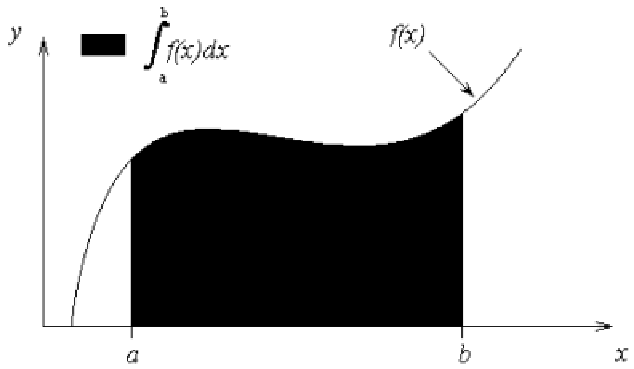


Figure 1 Integration of a function

Method 1:

Hence

$$I = \int_a^b f(x)dx \approx \int_a^b f_2(x)dx$$

where $f_2(x)$ is a second order polynomial given by

$$f_2(x) = a_0 + a_1 x + a_2 x^2.$$

Choose

$$(a, f(a)), \left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right), \text{ and } (b, f(b))$$

as the three points of the function to evaluate a_0, a_1 and a_2.

$$f(a) = f_2(a) = a_0 + a_1 a + a_2 a^2$$

$$f\left(\frac{a+b}{2}\right) = f_2\left(\frac{a+b}{2}\right) = a_0 + a_1\left(\frac{a+b}{2}\right) + a_2\left(\frac{a+b}{2}\right)^2$$

$$f(b) = f_2(b) = a_0 + a_1 b + a_2 b^2$$

Solving the above three equations for unknowns, a_0, a_1 and a_2 give

$$a_0 = \frac{a^2 f(b) + abf(b) - 4abf\left(\frac{a+b}{2}\right) + abf(a) + b^2 f(a)}{a^2 - 2ab + b^2}$$

$$a_1 = -\frac{af(a) - 4af\left(\frac{a+b}{2}\right) + 3af(b) + 3bf(a) - 4bf\left(\frac{a+b}{2}\right) + bf(b)}{a^2 - 2ab + b^2}$$

$$a_2 = \frac{2\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right)}{a^2 - 2ab + b^2}$$

Then

$$I \approx \int_a^b f_2(x)dx$$

$$= \int_a^b \left(a_0 + a_1 x + a_2 x^2\right)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3}\right]_a^b$$

$$= a_0(b-a) + a_1 \frac{b^2 - a^2}{2} + a_2 \frac{b^3 - a^3}{3}$$

Substituting values of a_0, a_1 and a_2 give

$$\int_a^b f_2(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Since for Simpson 1/3 rule, the interval $[a,b]$ is broken into 2 segments, the segment width

$$h = \frac{b-a}{2}$$

Since for Simpson 1/3 rule, the interval $[a, b]$ is broken into 2 segments, the segment width

$$h = \frac{b-a}{2}$$

Hence the Simpson's 1/3 rule is given by

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Since the above form has 1/3 in its formula, it is called Simpson's 1/3 rule.

<u>Method 2:</u>

Simpson's 1/3 rule can also be derived by approximating $f(x)$ by a second order polynomial using Newton's divided difference polynomial as

$$f_2(x) = b_0 + b_1(x-a) + b_2(x-a)\left(x - \frac{a+b}{2}\right)$$

where

$$b_0 = f(a)$$

$$b_1 = \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a}$$

$$b_2 = \frac{\dfrac{f(b) - f\left(\frac{a+b}{2}\right)}{b - \frac{a+b}{2}} - \dfrac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a}}{b-a}$$

Integrating Newton's divided difference polynomial gives us

$$\int_a^b f(x)dx \approx \int_a^b f_2(x)dx$$

$$= \int_a^b \left[b_0 + b_1(x-a) + b_2(x-a)\left(x - \frac{a+b}{2}\right)\right]dx$$

$$= \left[b_0 x + b_1\left(\frac{x^2}{2} - ax\right) + b_2\left(\frac{x^3}{3} - \frac{(3a+b)x^2}{4} + \frac{a(a+b)x}{2}\right)\right]_a^b$$

$$= b_0(b-a) + b_1\left(\frac{b^2 - a^2}{2} - a(b-a)\right)$$

$$+ b_2\left(\frac{b^3 - a^3}{3} - \frac{(3a+b)(b^2 - a^2)}{4} + \frac{a(a+b)(b-a)}{2}\right)$$

Substituting values of b_0, b_1, and b_2 into this equation yields the same result as before

$$\int_a^b f(x)dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Method 3:

One could even use the Lagrange polynomial to derive Simpson's formula. Notice any method of three-point quadratic interpolation can be used to accomplish this task. In this case, the interpolating function becomes

$$f_2(x)=\frac{\left(x-\frac{a+b}{2}\right)(x-b)}{\left(a-\frac{a+b}{2}\right)(a-b)}f(a)+\frac{(x-a)(x-b)}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)}f\left(\frac{a+b}{2}\right)+\frac{(x-a)\left(x-\frac{a+b}{2}\right)}{(b-a)\left(b-\frac{a+b}{2}\right)}f(b)$$

Integrating this function gets

$$\int_a^b f_2(x)dx=\left[\frac{\frac{x^3}{3}-\frac{(a+3b)x^2}{4}+\frac{b(a+b)x}{2}}{\left(a-\frac{a+b}{2}\right)(a-b)}f(a)+\frac{\frac{x^3}{3}-\frac{(a+b)x^2}{2}+abx}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)}f\left(\frac{a+b}{2}\right)+\frac{\frac{x^3}{3}-\frac{(3a+b)x^2}{4}+\frac{a(a+b)x}{2}}{(b-a)\left(b-\frac{a+b}{2}\right)}f(b)\right]_a^b$$

$$=\frac{\frac{b^3-a^3}{3}-\frac{(a+3b)(b^2-a^2)}{4}+\frac{b(a+b)(b-a)}{2}}{\left(a-\frac{a+b}{2}\right)(a-b)}f(a)$$

$$+\frac{\frac{b^3-a^3}{3}-\frac{(a+b)(b^2-a^2)}{2}+ab(b-a)}{\left(\frac{a+b}{2}-a\right)\left(\frac{a+b}{2}-b\right)}f\left(\frac{a+b}{2}\right)$$

$$+\frac{\frac{b^3-a^3}{3}-\frac{(3a+b)(b^2-a^2)}{4}+\frac{a(a+b)(b-a)}{2}}{(b-a)\left(b-\frac{a+b}{2}\right)}f(b)$$

Believe it or not, simplifying and factoring this large expression yields you the same result as before

$$\int_a^b f(x)dx\approx\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

$$=\frac{h}{3}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right].$$

Method 4:
Simpson's 1/3 rule can also be derived by the method of coefficients. Assume

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f\left(\frac{a+b}{2}\right) + c_3 f(b)$$

Let the right-hand side be an exact expression for the integrals $\int_a^b 1dx$, $\int_a^b xdx$, and $\int_a^b x^2 dx$. This implies that the right hand side will be exact expressions for integrals of any linear combination of the three integrals for a general second order polynomial. Now

$$\int_a^b 1dx = b - a = c_1 + c_2 + c_3$$

$$\int_a^b xdx = \frac{b^2 - a^2}{2} = c_1 a + c_2 \frac{a+b}{2} + c_3 b$$

$$\int_a^b x^2 dx = \frac{b^3 - a^3}{3} = c_1 a^2 + c_2 \left(\frac{a+b}{2}\right)^2 + c_3 b^2$$

Solving the above three equations for c_0, c_1 and c_2 give

$$c_1 = \frac{b-a}{6}$$

$$c_2 = \frac{2(b-a)}{3}$$

$$c_3 = \frac{b-a}{6}$$

This gives

$$\int_a^b f(x)dx \approx \frac{b-a}{6} f(a) + \frac{2(b-a)}{3} f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} f(b)$$

$$= \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

The integral from the first method

$$\int_a^b f(x)dx \approx \int_a^b (a_0 + a_1 x + a_2 x^2)dx$$

can be viewed as the area under the second order polynomial, while the equation from Method 4

$$\int_a^b f(x)dx \approx \frac{b-a}{6} f(a) + \frac{2(b-a)}{3} f\left(\frac{a+b}{2}\right) + \frac{b-a}{6} f(b)$$

can be viewed as the sum of the areas of three rectangles.

Example 1

The distance covered by a rocket in meters from $t = 8s$ to $t = 30s$ is given by

$$x = \int_8^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

a) Use Simpson's 1/3 rule to find the approximate value of x.

b) Find the true error, E_t.

c) Find the absolute relative true error, $|\in_t|$.

Solution

a) $$x \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

$a = 8$

$b = 30$

$\frac{a+b}{2} = 19$

$$f(t) = 2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t$$

$$f(8) = 2000 \ln \left[\frac{140000}{140000 - 2100(8)} \right] - 9.8(8) = 177.27 m/s$$

$$f(30) = 2000 \ln \left[\frac{140000}{140000 - 2100(30)} \right] - 9.8(30) = 901.67 m/s$$

$$f(19) = 2000 \ln \left(\frac{140000}{140000 - 2100(19)} \right) - 9.8(19) = 484.75 m/s$$

$$x \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

$$= \left(\frac{30-8}{6} \right) [f(8) + 4f(19) + f(30)]$$

$$= \frac{22}{6}[177.27 + 4 \times 484.75 + 901.67]$$

=11065.72 m

b) The exact value of the above integral is

$$x = \int_{8}^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

=11061.34 m

So the true error is

$$E_t = True\ Value - Approximate\ Value$$

=11061.34-11065.72

$= -4.38\ m$

c) The absolute relative true error is

$$|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100$$

$$= \left|\frac{-4.38}{11061.34}\right| \times 100$$

Multiple-segment Simpson's 1/3 Rule

Just like in multiple-segment trapezoidal rule, one can subdivide the interval $[a,b]$ into n segments and apply Simpson's 1/3 rule repeatedly over every two segments. Note that n needs to be even. Divide interval $[a,b]$ into n equal segments, so that the segment width is given by

$$h = \frac{b-a}{n}.$$

Now

$$\int_a^b f(x)dx = \int_{x_0}^{x_n} f(x)dx$$

where

$$x_0 = a$$

$$x_n = b$$

$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots\ldots + \int_{x_{n-4}}^{x_{n-2}} f(x)dx + \int_{x_{n-2}}^{x_n} f(x)dx$$

Apply Simpson's 1/3rd Rule over each interval,

$$\int_a^b f(x)dx \cong (x_2 - x_0)\left[\frac{f(x_0)+4f(x_1)+f(x_2)}{6}\right] + (x_4 - x_2)\left[\frac{f(x_2)+4f(x_3)+f(x_4)}{6}\right] + \ldots$$

$$+ (x_{n-2} - x_{n-4})\left[\frac{f(x_{n-4})+4f(x_{n-3})+f(x_{n-2})}{6}\right] + (x_n - x_{n-2})\left[\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\right]$$

$$\int_a^b f(x)dx \cong 2h\left[\frac{f(x_0)+4f(x_1)+f(x_2)}{6}\right] + 2h\left[\frac{f(x_2)+4f(x_3)+f(x_4)}{6}\right] + \ldots$$

$$+ 2h\left[\frac{f(x_{n-4})+4f(x_{n-3})+f(x_{n-2})}{6}\right] + 2h\left[\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4\{f(x_1)+f(x_3)+\ldots+f(x_{n-1})\} + 2\{f(x_2)+f(x_4)+\ldots+f(x_{n-2})\} + f(x_n)\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(x_i) + f(x_n) \right]$$

$$\boxed{\int_a^b f(x)dx \cong \frac{b-a}{3n}\left[f(x_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(x_i) + f(x_n) \right]}$$

Example 2

Use 4-segment Simpson's 1/3 rule to approximate the distance covered by a rocket in meters from $t = 8$ s to $t = 30$ s as given by

$$x = \int_8^{30} \left(2000 \ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

a) Use four segment Simpson's 1/3rd Rule to estimate x.
b) Find the true error, E_t for part (a).
c) Find the absolute relative true error, $|\in_t|$ for part (a).

Solution:

a) Using n segment Simpson's 1/3 rule,

$$x \approx \frac{b-a}{3n}\left[f(t_0) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f(t_i) + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f(t_i) + f(t_n) \right]$$

$$n = 4$$
$$a = 8$$
$$b = 30$$
$$h = \frac{b-a}{n}$$
$$= \frac{30-8}{4}$$
$$= 5.5$$
$$f(t) = 2000 \ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t$$

So

$$f(t_0) = f(8)$$
$$f(8) = 2000 \ln\left[\frac{140000}{140000 - 2100(8)} \right] - 9.8(8) = 177.27 m/s$$

$$f(t_1) = f(8 + 5.5) = f(13.5)$$

$$f(13.5)=2000\ln\left[\frac{140000}{140000-2100(13.5)}\right]-9.8(13.5)=320.25m/s$$

$$f(t_2)=f(13.5+5.5)=f(19)$$

$$f(19)=2000\ln\left(\frac{140000}{140000-2100(19)}\right)-9.8(19)=484.75m/s$$

$$f(t_3)=f(19+5.5)=f(24.5)$$

$$f(24.5)=2000\ln\left[\frac{140000}{140000-2100(24.5)}\right]-9.8(24.5)=676.05m/s$$

$$f(t_4)=f(t_n)=f(30)$$

$$f(30)=2000\ln\left[\frac{140000}{140000-2100(30)}\right]-9.8(30)=901.67m/s$$

$$\begin{aligned}
x&=\frac{b-a}{3n}\left[f(t_0)+4\sum_{\substack{i=1\\i=odd}}^{n-1}f(t_i)+2\sum_{\substack{i=2\\i=even}}^{n-2}f(t_i)+f(t_n)\right]\\
&=\frac{30-8}{3(4)}\left[f(8)+4\sum_{\substack{i=1\\i=odd}}^{3}f(t_i)+2\sum_{\substack{i=2\\i=even}}^{2}f(t_i)+f(30)\right]\\
&=\frac{22}{12}[f(8)+4f(t_1)+4f(t_3)+2f(t_2)+f(30)]\\
&=\frac{11}{6}[f(8)+4f(13.5)+4f(24.5)+2f(19)+f(30)]\\
&=\frac{11}{6}[177.27+4(320.25)+4(676.05)+2(484.75)+901.67]\\
&=11061.64\ m
\end{aligned}$$

b) The exact value of the above integral is

$$x=\int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

=11061.34 m

So the true error is

$$E_t=True\ Value-Approximate\ Value$$

$$E_t=11061.34-11061.64$$

$$=-0.30\ m$$

c) The absolute relative true error is

$$|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100$$

$$= \left|\frac{-0.3}{11061.34}\right| \times 100$$

$$= 0.0027\%$$

Table 1 Values of Simpson's 1/3 rule for Example 2 with multiple-segments

n	Approximate Value	E_t	$\lvert\in_t\rvert$
2	11065.72	-4.38	0.0396%
4	11061.64	-0.30	0.0027%
6	11061.40	-0.06	0.0005%
8	11061.35	-0.02	0.0002%
10	11061.34	-0.01	0.0001%

Error in Multiple-segment Simpson's 1/3 rule

The true error in a single application of Simpson's 1/3rd Rule is given[1] by

$$E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\zeta), \quad a < \zeta < b$$

In multiple-segment Simpson's 1/3 rule, the error is the sum of the errors in each application of Simpson's 1/3 rule. The error in the n segments Simpson's 1/3rd Rule is given by

$$E_1 = -\frac{(x_2 - x_0)^5}{2880} f^{(4)}(\zeta_1), \quad x_0 < \zeta_1 < x_2$$

$$= -\frac{h^5}{90} f^{(4)}(\zeta_1)$$

$$E_{\frac{n}{2}-1} = -\frac{(x_{n-2} - x_{n-4})^5}{2880} f^{(4)}\left(\zeta_{\frac{n}{2}-1}\right), \quad x_{n-4} < \zeta_{\frac{n}{2}-1} < x_{n-2}$$

$$= -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}-1}\right)$$

$$E_{\frac{n}{2}} = -\frac{(x_n - x_{n-2})^5}{2880} f^{(4)}\left(\zeta_{\frac{n}{2}}\right), x_{n-2} < \zeta_{\frac{n}{2}} < x_n$$

Hence, the total error in the multiple-segment Simpson's 1/3 rule is

$$= -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}}\right)$$

$$E_t = \sum_{i=1}^{\frac{n}{2}} E_i$$

$$= -\frac{h^5}{90} \sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)$$

$$= -\frac{(b-a)^5}{90n^5} \sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)$$

$$= -\frac{(b-a)^5}{180n^4} \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{\frac{n}{2}},$$

Simpsons 3/8 Rule for Integration

Substituting the form of $f_3(x)$ from Method (1) or Method (2),

$$I = \int_a^b f(x)dx$$
$$\approx \int_a^b f_3(x)dx$$
$$= (b-a)\times \frac{\{f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\}}{8} \qquad (11)$$

Since

$$h = \frac{b-a}{3}$$
$$b-a = 3h$$

and Equation (11) becomes

$$I \approx \frac{3h}{8}\times\{f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\} \qquad (12)$$

Note the 3/8 in the formula, and hence the name of method as the Simpson's 3/8 rule.
The true error in Simpson 3/8 rule can be derived as [Ref. 1]

$$E_t = -\frac{(b-a)^5}{6480}\times f''''(\zeta)\text{, where } a \le \zeta \le b \qquad (13)$$

Example 1

The vertical distance in meters covered by a rocket from $t = 8$ to $t = 30$ seconds is given by

$$s = \int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

Use Simpson 3/8 rule to find the approximate value of the integral.

Solution

$$h = \frac{b-a}{n}$$
$$= \frac{b-a}{3}$$
$$= \frac{30-8}{3}$$
$$= 7.3333$$

$$f(t) = 2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t$$

$$I \approx \frac{3h}{8}\times\{f(t_0)+3f(t_1)+3f(t_2)+f(t_3)\}$$

$$t_0 = 8$$

$$f(t_0) = 2000\ln\left(\frac{140000}{140000-2100\times 8}\right)-9.8\times 8$$
$$= 177.2667$$

$$
\begin{cases}
t_1 = t_0 + h \\
\quad = 8 + 7.3333 \\
\quad = 15.3333 \\
f(t_1) = 2000\ln\left(\dfrac{140000}{140000 - 2100 \times 15.3333}\right) - 9.8 \times 15.3333 \\
\quad = 372.4629
\end{cases}
$$

$$
\begin{cases}
t_2 = t_0 + 2h \\
\quad = 8 + 2(7.3333) \\
\quad = 22.6666 \\
f(t_2) = 2000\ln\left(\dfrac{140000}{140000 - 2100 \times 22.6666}\right) - 9.8 \times 22.6666 \\
\quad = 608.8976
\end{cases}
$$

$$
\begin{cases}
t_3 = t_0 + 3h \\
\quad = 8 + 3(7.3333) \\
\quad = 30 \\
f(t_3) = 2000\ln\left(\dfrac{140000}{140000 - 2100 \times 30}\right) - 9.8 \times 30 \\
\quad = 901.6740
\end{cases}
$$

Applying Equation (12), one has

$$
\begin{aligned}
I &= \frac{3}{8} \times 7.3333 \times \{177.2667 + 3 \times 372.4629 + 3 \times 608.8976 + 901.6740\} \\
&= 11063.3104\, m
\end{aligned}
$$

The exact answer can be computed as

$$
I_{exact} = 11061.34\, m
$$

Multiple Segments for Simpson 3/8 Rule

Using n = number of equal segments, the width h can be defined as

$$
h = \frac{b-a}{n} \qquad (14)
$$

The number of segments need to be an integer multiple of 3 as a single application of Simpson 3/8 rule requires 3 segments.

The integral shown in Equation (1) can be expressed as

$$
\begin{aligned}
I &= \int_a^b f(x)dx \\
&\approx \int_a^b f_3(x)dx \\
&\approx \int_{x_0=a}^{x_3} f_3(x)dx + \int_{x_3}^{x_6} f_3(x)dx + \ldots\ldots + \int_{x_{n-3}}^{x_n=b} f_3(x)dx
\end{aligned} \tag{15}
$$

Using Simpson 3/8 rule (See Equation 12) into Equation (15), one gets

$$
I = \frac{3h}{8}\begin{Bmatrix} f(x_0)+3f(x_1)+3f(x_2)+f(x_3)+f(x_3)+3f(x_4)+3f(x_5)+f(x_6) \\ +\ldots..+f(x_{n-3})+3f(x_{n-2})+3f(x_{n-1})+f(x_n) \end{Bmatrix} \tag{16}
$$

$$
= \frac{3h}{8}\left\{ f(x_0)+3\sum_{i=1,4,7,..}^{n-2} f(x_i)+3\sum_{i=2,5,8,..}^{n-1} f(x_i)+2\sum_{i=3,6,9,..}^{n-3} f(x_i)+f(x_n)\right\} \tag{17}
$$

Example 2

The vertical distance in meters covered by a rocket from $t=8$ to $t=30$ seconds is given by

$$
s = \int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt
$$

Use Simpson 3/8 multiple segments rule with six segments to estimate the vertical distance.

Solution

In this example, one has (see Equation 14):

$$
f(t) = 2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t
$$

$$
h = \frac{30-8}{6} = 3.6666
$$

$\{t_0, f(t_0)\} = \{8,177.2667\}$

$\{t_1, f(t_1)\} = \{11.6666,270.4104\}$ where $t_1 = t_0 + h = 8 + 3.6666 = 11.6666$

$\{t_2, f(t_2)\} = \{15.3333,372.4629\}$ where $t_2 = t_0 + 2h = 15.3333$

$\{t_3, f(t_3)\} = \{19,484.7455\}$ where $t_3 = t_0 + 3h = 19$

$\{t_4, f(t_4)\} = \{22.6666,608.8976\}$ where $t_4 = t_0 + 4h = 22.6666$

$\{t_5, f(t_5)\} = \{26.3333,746.9870\}$ where $t_5 = t_0 + 5h = 26.3333$

$\{t_6, f(t_6)\} = \{30,901.6740\}$ where $t_6 = t_0 + 6h = 30$

Applying Equation (17), one obtains:

$$
I = \frac{3}{8}(3.6666)\left\{177.2667+3\sum_{i=1,4,..}^{n-2=4} f(t_i)+3\sum_{i=2,5,..}^{n-1=5} f(t_i)+2\sum_{i=3,6,..}^{n-3=3} f(t_i)+901.6740\right\}
$$

$$
= (1.3750)\begin{Bmatrix} 177.2667+3(270.4104+608.8976) \\ +3(372.4629+746.9870)+2(484.7455)+901.6740 \end{Bmatrix}
$$

$$
= 11{,}601.4696\,m
$$

What is Euler's method?

Euler's method is a numerical technique to solve ordinary differential equations of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0 \tag{1}$$

So only first order ordinary differential equations can be solved by using Euler's method. In another chapter we will discuss how Euler's method is used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations. How does one write a first order differential equation in the above form?

Example 1

Rewrite

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y),\ \ y(0) = y_0 \text{ form.}$$

Solution

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example 2

Rewrite

$$e^y \frac{dy}{dx} + x^2 y^2 = 2\sin(3x),\ \ y(0) = 5$$

in

$$\frac{dy}{dx} = f(x, y),\ \ y(0) = y_0 \text{ form.}$$

Solution

$$e^y \frac{dy}{dx} + x^2 y^2 = 2\sin(3x),\ \ y(0) = 5$$

$$\frac{dy}{dx} = \frac{2\sin(3x) - x^2 y^2}{e^y},\ \ y(0) = 5$$

In this case

$$f(x, y) = \frac{2\sin(3x) - x^2 y^2}{e^y}$$

Derivation of Euler's method

At $x = 0$, we are given the value of $y = y_0$. Let us call $x = 0$ as x_0. Now since we know the slope of y with respect to x, that is, $f(x, y)$, then at $x = x_0$, the slope is $f(x_0, y_0)$. Both x_0 and y_0 are known from the initial condition $y(x_0) = y_0$.

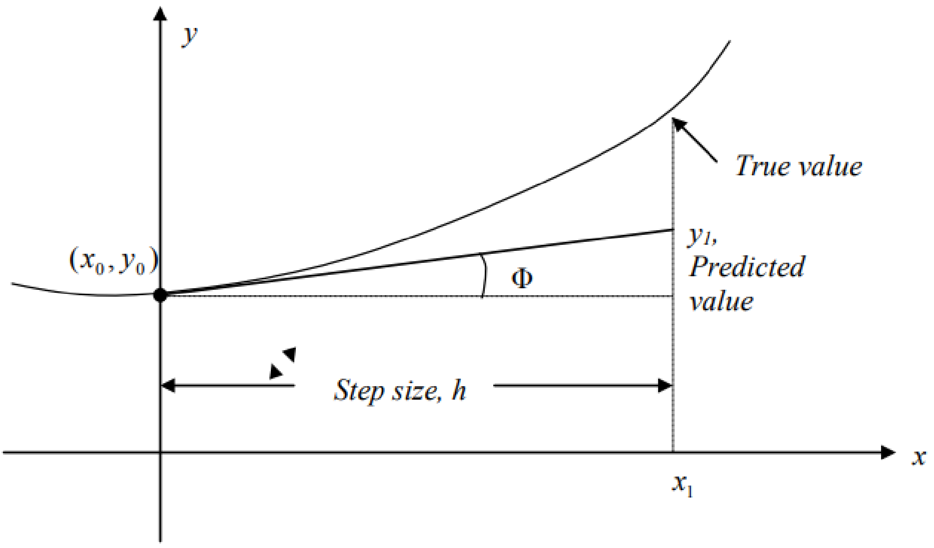


Figure 1 Graphical interpretation of the first step of Euler's method.

So the slope at $x = x_0$ as shown in Figure 1 is

$$\begin{aligned}\text{Slope} &= \frac{Rise}{Run} \\ &= \frac{y_1 - y_0}{x_1 - x_0} \\ &= f(x_0, y_0)\end{aligned}$$

From here

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

Calling $x_1 - x_0$ the step size h, we get

$$y_1 = y_0 + f(x_0, y_0)h \qquad (2)$$

One can now use the value of y_1 (an approximate value of y at $x = x_1$) to calculate y_2, and that would be the predicted value at x_2, given by

$$y_2 = y_1 + f(x_1, y_1)h$$
$$x_2 = x_1 + h$$

Based on the above equations, if we now know the value of $y = y_i$ at x_i, then

$$y_{i+1} = y_i + f(x_i, y_i)h \qquad (3)$$

This formula is known as Euler's method and is illustrated graphically in Figure 2. In some books, it is also called the Euler-Cauchy method.

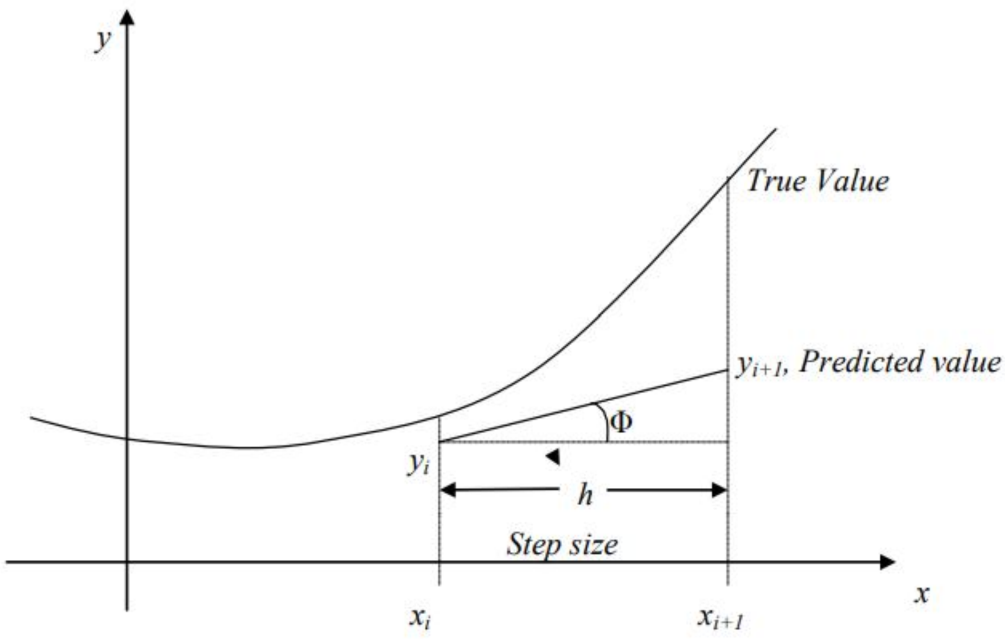


Figure 2 General graphical interpretation of Euler's method.

Find an approximate value of

$$\int_5^8 6x^3 dx$$

using Euler's method of solving an ordinary differential equation. Use a step size of $h = 1.5$.

Solution

Given $\int_5^8 6x^3 dx$, we can rewrite the integral as the solution of an ordinary differential equation

$$\frac{dy}{dx} = 6x^3, \ y(5) = 0$$

where $y(8)$ will give the value of the integral $\int_5^8 6x^3 dx$.

$$\frac{dy}{dx} = 6x^3 = f(x, y), \ y(5) = 0$$

The Euler's method equation is

$$y_{i+1} = y_i + f(x_i, y_i)h$$

<u>Step 1</u>

$$i = 0, \ x_0 = 5, \ y_0 = 0$$

$$h = 1.5$$

$$\begin{aligned} x_1 &= x_0 + h \\ &= 5 + 1.5 \\ &= 6.5 \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + f(x_0, y_0)h \\ &= 0 + f(5,0) \times 1.5 \\ &= 0 + \left(6 \times 5^3\right) \times 1.5 \\ &= 1125 \\ &\approx y(6.5) \end{aligned}$$

Step 2

$$i = 1,\ x_1 = 6.5,\ y_1 = 1125$$

$$\begin{aligned} x_2 &= x_1 + h \\ &= 6.5 + 1.5 \\ &= 8 \\ y_2 &= y_1 + f(x_1, y_1)h \\ &= 1125 + f(6.5, 1125) \times 1.5 \\ &= 1125 + (6 \times 6.5^3) \times 1.5 \\ &= 3596.625 \\ &\approx y(8) \end{aligned}$$

Hence

$$\begin{aligned} \int_5^8 6x^3 dx &= y(8) - y(5) \\ &\approx 3596.625 - 0 \\ &= 3596.625 \end{aligned}$$