Limit Theorems

We shall now obtain results that are useful in calculating limits of functions. These resul are parallel to the limit theorems established in Section 3.2 for sequences. In fact, in mo cases these results can be proved by using Theorem 4.1.8 and results from Section 3.: Alternatively, the results in this section can be proved by using ε-δ arguments that are ve similar to the ones employed in Section 3.2.

4.2.1 Definition Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. We say that f is **bounded on a neighborhood of** c if there exists a δ-neighborhood $V_\delta(c)$ of c and a constant $M > 0$ such that we have $|f(x)| \le M$ for all $x \in A \cap V_\delta(c)$.

4.2.2 Theorem *If $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ has a limit at $c \in \mathbb{R}$, then f is bounded on some neighborhood of c.*

Proof. If $L := \lim_{x \to c} f$, then for $\varepsilon = 1$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < 1$; hence (by Corollary 2.2.4(a)),

$$|f(x)| - |L| \le |f(x) - L| < 1.$$

Therefore, if $x \in A \cap V_\delta(c)$, $x \ne c$, then $|f(x)| \le |L| + 1$. If $c \notin A$, we take $M = |L| + 1$, while if $c \in A$ we take $M := \sup\{|f(c)|, |L| + 1\}$. It follows that if $x \in A \cap V_\delta(c)$, then $|f(x)| \le M$. This shows that f is bounded on the neighborhood $V_\delta(c)$ of c. Q.E.D.

The next definition is similar to the definition for sums, differences, products, and quotients of sequences given in Section 3.2.

4.2.3 Definition Let $A \subseteq \mathbb{R}$ and let f and g be functions defined on A to $\mathbb{R}$. We define the **sum** $f + g$, the **difference** $f - g$, and the **product** fg on A to $\mathbb{R}$ to be the functions given by

$$(f+g)(x) := f(x) + g(x), \qquad (f-g)(x) := f(x) - g(x),$$
$$(fg)(x) := f(x)g(x)$$

for all $x \in A$. Further, if $b \in \mathbb{R}$, we define the **multiple** bf to be the function given by

$$(bf)(x) := bf(x) \quad \text{for all} \quad x \in A.$$

Finally, if $h(x) \ne 0$ for $x \in A$, we define the **quotient** f/h to be the function given by

$$\left(\frac{f}{h}\right)(x) := \frac{f(x)}{h(x)} \qquad \text{for all} \qquad x \in A.$$

4.2.4 Theorem *Let $A \subseteq \mathbb{R}$, let f and g be functions on A to $\mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. Further, let $b \in \mathbb{R}$.*

(a) *If $\lim_{x \to c} f = L$ and $\lim_{x \to c} g = M$, then:*

$$\lim_{x \to c} (f + g) = L + M, \qquad \lim_{x \to c} (f - g) = L - M,$$
$$\lim_{x \to c} (fg) = LM, \qquad \lim_{x \to c} (bf) = bL.$$

(b) *If $h : A \to \mathbb{R}$, if $h(x) \ne 0$ for all $x \in A$, and if $\lim_{x \to c} h = H \ne 0$, then*

$$\lim_{x \to c} \left(\frac{f}{h}\right) = \frac{L}{H}.$$

Proof. One proof of this theorem is exactly similar to that of Theorem 3.2.3. Alternatively, it can be proved by making use of Theorems 3.2.3 and 4.1.8. For example, let (x_n) be any sequence in A such that $x_n \neq c$ for $n \in \mathbb{N}$, and $c = \lim(x_n)$. It follows from Theorem 4.1.8 that

$$\lim(f(x_n)) = L, \qquad \lim(g(x_n)) = M.$$

On the other hand, Definition 4.2.3 implies that

$$(fg)(x_n) = f(x_n)g(x_n) \qquad \text{for} \quad n \in \mathbb{N}.$$

Therefore an application of Theorem 3.2.3 yields

$$\begin{aligned} \lim((fg)(x_n)) &= \lim(f(x_n)g(x_n)) \\ &= [\lim(f(x_n))]\,[\lim(g(x_n))] = LM. \end{aligned}$$

Consequently, it follows from Theorem 4.1.8 that

$$\lim_{x \to c}(fg) = \lim((fg)(x_n)) = LM.$$

The other parts of this theorem are proved in a similar manner. We leave the details to the reader. Q.E.D.

Remark Let $A \subseteq \mathbb{R}$, and let $f_1, f_2, \ldots, f_n$ be functions on A to $\mathbb{R}$, and let c be a cluster point of A. If $L_k := \lim_{x \to c} f_k$ for $k = 1, \ldots, n$, then it follows from Theorem 4.2.4 by an Induction argument that

$$L_1 + L_2 + \cdots + L_n = \lim_{x \to c}(f_1 + f_2 + \cdots + f_n),$$

and

$$L_1 \cdot L_2 \cdots L_n = \lim(f_1 \cdot f_2 \cdots f_n).$$

In particular, we deduce that if $L = \lim_{x \to c} f$ and $n \in \mathbb{N}$, then

$$L^n = \lim_{x \to c}(f(x))^n.$$

4.2.5 Examples **(a)** Some of the limits that were established in Section 4.1 can be proved by using Theorem 4.2.4. For example, it follows from this result that since $\lim_{x \to c} x = c$, then $\lim_{x \to c} x^2 = c^2$, and that if $c > 0$, then

$$\lim_{x \to c} \frac{1}{x} = \frac{1}{\lim_{x \to c} x} = \frac{1}{c}.$$

(b) $\lim_{x \to 2}(x^2 + 1)(x^3 - 4) = 20.$

It follows from Theorem 4.2.4 that

$$\begin{aligned} \lim_{x \to 2}(x^2 + 1)(x^3 - 4) &= \left(\lim_{x \to 2}(x^2 + 1)\right)\left(\lim_{x \to 2}(x^3 - 4)\right) \\ &= 5 \cdot 4 = 20. \end{aligned}$$

If we apply Theorem 4.2.4(b), we have

$$\lim_{x\to 2}\frac{x^3-4}{x^2+1}=\frac{\lim\limits_{x\to 2}\left(x^3-4\right)}{\lim\limits_{x\to 2}\left(x^2+1\right)}=\frac{4}{5}.$$

Note that since the limit in the denominator [i.e., $\lim\limits_{x\to 2}\left(x^2+1\right)=5$] is not equal to 0, then Theorem 4.2.4(b) is applicable.

(d) $\lim\limits_{x\to 2}=\dfrac{x^2-4}{3x-6}=\dfrac{4}{3}$.

If we let $f(x):=x^2-4$ and $h(x):=3x-6$ for $x\in\mathbb{R}$, then we *cannot* use Theorem 4.2.4(b) to evaluate $\lim\limits_{x\to 2}(f(x)/h(x))$ because

$$H=\lim_{x\to 2}h(x)=\lim_{x\to 2}(3x-6)=3\cdot 2-6=0.$$

However, if $x\neq 2$, then it follows that

$$\frac{x^2-4}{3x-6}=\frac{(x+2)(x-2)}{3(x-2)}=\frac{1}{3}(x+2).$$

Therefore we have

$$\lim_{x\to 2}\frac{x^2-4}{3x-6}=\lim_{x\to 2}\frac{1}{3}(x+2)=\frac{1}{3}\left(\lim_{x\to 2}x+2\right)=\frac{4}{3}.$$

Note that the function $g(x)=(x^2-4)/(3x-6)$ has a limit at $x=2$ *even though it is not defined there.*

(e) $\lim\limits_{x\to 0}\dfrac{1}{x}$ does not exist in $\mathbb{R}$.

Of course $\lim\limits_{x\to 0}1=1$ and $H:=\lim\limits_{x\to 0}x=0$. However, since $H=0$, we *cannot* use Theorem 4.2.4(b) to evaluate $\lim\limits_{x\to 0}(1/x)$. In fact, as was seen in Example 4.1.10(a), the function $\varphi(x)=1/x$ does not have a limit at $x=0$. This conclusion also follows from Theorem 4.2.2 since the function $\varphi(x)=1/x$ is not bounded on a neighborhood of $x=0$.

(f) If p is a polynomial function, then $\lim\limits_{x\to c}p(x)=p(c)$.

Let p be a polynomial function on $\mathbb{R}$ so that $p(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ for all $x\in\mathbb{R}$. It follows from Theorem 4.2.4 and the fact that $\lim\limits_{x\to c}x^k=c^k$ that

$$\begin{aligned}\lim_{x\to c}p(x) &= \lim_{x\to c}\left[a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0\right]\\ &= \lim_{x\to c}(a_nx^n)+\lim_{x\to c}\left(a_{n-1}x^{n-1}\right)+\cdots+\lim_{x\to c}(a_1x)+\lim_{x\to c}a_0\\ &= a_nc^n+a_{n-1}c^{n-1}+\cdots+a_1c+a_0\\ &= p(c).\end{aligned}$$

Hence $\lim\limits_{x\to c}p(x)=p(c)$ for any polynomial function p.

(g) If p and q are polynomial functions on $\mathbb{R}$ and if $q(c) \neq 0$, then

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}.$$

Since $q(x)$ is a polynomial function, it follows from a theorem in algebra that there are at most a finite number of real numbers $\alpha_1, \ldots, \alpha_m$ [the real zeroes of $q(x)$] such that $q(\alpha_j) = 0$ and such that if $x \notin \{\alpha_1, \ldots, \alpha_m\}$, then $q(x) \neq 0$. Hence, if $x \notin \{\alpha_1, \ldots, \alpha_m\}$, we can define

$$r(x) := \frac{p(x)}{q(x)}.$$

If c is not a zero of $q(x)$, then $q(c) \neq 0$, and it follows from part (f) that $\lim_{x \to c} q(x) = q(c) \neq 0$. Therefore we can apply Theorem 4.2.4(b) to conclude that

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{\lim_{x \to c} p(x)}{\lim_{x \to c} q(x)} = \frac{p(c)}{q(c)}. \qquad \square$$

The next result is a direct analogue of Theorem 3.2.6.

4.2.6 Theorem *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. If*

$$a \leq f(x) \leq b \quad \text{for all} \quad x \in A, x \neq c,$$

and if $\lim_{x \to c} f$ exists, then $a \leq \lim_{x \to c} f \leq b$.

Proof. Indeed, if $L = \lim_{x \to c} f$, then it follows from Theorem 4.1.8 that if (x_n) is any sequence of real numbers such that $c \neq x_n \in A$ for all $n \in \mathbb{N}$ and if the sequence (x_n) converges to c, then the sequence $(f(x_n))$ converges to L. Since $a \leq f(x_n) \leq b$ for all $n \in \mathbb{N}$, it follows from Theorem 3.2.6 that $a \leq L \leq b$. Q.E.D.

We now state an analogue of the Squeeze Theorem 3.2.7. We leave its proof to the reader.

4.2.7 Squeeze Theorem *Let $A \subseteq \mathbb{R}$, let $f, g, h : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. If*

$$f(x) \leq g(x) \leq h(x) \quad \text{for all} \quad x \in A, \ x \neq c,$$

and if $\lim_{x \to c} f = L = \lim_{x \to c} h$, then $\lim_{x \to c} g = L$.

4.2.7 Squeeze Theorem *Let $A \subseteq \mathbb{R}$, let $f, g, h\colon A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. If*

$$f(x) \le g(x) \le h(x) \quad \textit{for all} \quad x \in A,\ x \ne c,$$

and if $\lim_{x\to c} f = L = \lim_{x\to c} h$, *then* $\lim_{x\to c} g = L$.

4.2.8 Examples **(a)** $\lim_{x\to 0} x^{3/2} = 0\,(x > 0)$.

Let $f(x) := x^{3/2}$ for $x > 0$. Since the inequality $x < x^{1/2} \le 1$ holds for $0 < x \le 1$ (why?), it follows that $x^2 \le f(x) = x^{3/2} \le x$ for $0 < x \le 1$. Since

$$\lim_{x\to 0} x^2 = 0 \quad \text{and} \quad \lim_{x\to 0} x = 0,$$

it follows from the Squeeze Theorem 4.2.7 that $\lim_{x\to 0} x^{3/2} = 0$.

(b) $\lim_{x\to 0} \sin x = 0$.

It will be proved later (see Theorem 8.4.8), that

$$-x \le \sin x \le x \quad \text{for all} \quad x \ge 0\,.$$

Since $\lim_{x\to 0}(\pm x) = 0$, it follows from the Squeeze Theorem that $\lim_{x\to 0} \sin x = 0$.

(c) $\lim_{x\to 0} \cos x = 1$.

It will be proved later (see Theorem 8.4.8) that

(1) $$1 - \tfrac{1}{2}x^2 \le \cos x \le 1 \quad \text{for all} \quad x \in \mathbb{R}.$$

Since $\lim_{x\to 0}\left(1 - \frac{1}{2}x^2\right) = 1$, it follows from the Squeeze Theorem that $\lim_{x\to 0} \cos x = 1$.

(d) $\lim_{x\to 0}\left(\dfrac{\cos x - 1}{x}\right) = 0$.

We cannot use Theorem 4.2.4(b) to evaluate this limit. (Why not?) However, it follows from the inequality (1) in part (c) that

$$-\tfrac{1}{2}x \leq (\cos x - 1)/x \leq 0 \quad \text{for} \quad x > 0$$

and that

$$0 \leq (\cos x - 1)/x \leq -\tfrac{1}{2}x \quad \text{for} \quad x < 0.$$

Now let $f(x) := -x/2$ for $x \geq 0$ and $f(x) := 0$ for $x < 0$, and let $h(x) := 0$ for $x \geq 0$ and $h(x) := -x/2$ for $x < 0$. Then we have

$$f(x) \leq (\cos x - 1)/x \leq h(x) \quad \text{for} \quad x \neq 0.$$

Since it is readily seen that $\lim\limits_{x\to 0} f = 0 = \lim\limits_{x\to 0} h$, it follows from the Squeeze Theorem that $\lim\limits_{x\to 0} (\cos x - 1)/x = 0$.

(e) $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right) = 1.$

Again we cannot use Theorem 4.2.4(b) to evaluate this limit. However, it will be proved later (see Theorem 8.4.8) that

$$x - \tfrac{1}{6}x^3 \leq \sin x \leq x \quad \text{for} \quad x \geq 0$$

and that

$$x \leq \sin x \leq x - \tfrac{1}{6}x^3 \quad \text{for} \quad x \leq 0.$$

Therefore it follows (why?) that

$$1 - \tfrac{1}{6}x^2 \leq (\sin x)/x \leq 1 \quad \text{for all} \quad x \neq 0.$$

But since $\lim\limits_{x\to 0}\left(1 - \tfrac{1}{6}x^2\right) = 1 - \tfrac{1}{6}\cdot \lim\limits_{x\to 0} x^2 = 1$, we infer from the Squeeze Theorem that $\lim\limits_{x\to 0} (\sin x)/x = 1$.

(f) $\lim_{x\to 0}(x\sin(1/x)) = 0$.

Let $f(x) = x\sin(1/x)$ for $x \neq 0$. Since $-1 \le \sin z \le 1$ for all $z \in \mathbb{R}$, we have the inequality

$$-|x| \le f(x) = x\sin(1/x) \le |x|$$

for all $x \in \mathbb{R}, x \neq 0$. Since $\lim_{x\to 0}|x| = 0$, it follows from the Squeeze Theorem that $\lim_{x\to 0} f = 0$. For a graph, see Figure 5.1.3 or the cover of this book. □

There are results that are parallel to Theorems 3.2.9 and 3.2.10; however, we will leave them as exercises. We conclude this section with a result that is, in some sense, a partial converse to Theorem 4.2.6.

4.2.9 Theorem *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A. If*

$$\lim_{x\to c} f > 0 \quad \left[\textit{respectively, } \lim_{x\to c} f < 0\right],$$

then there exists a neighborhood $V_\delta(c)$ of c such that $f(x) > 0$ [respectively, $f(x) < 0$] for all $x \in A \cap V_\delta(c)$, $x \neq c$.

Proof. Let $L := \lim_{x\to c} f$ and suppose that $L > 0$. We take $\varepsilon = \frac{1}{2}L > 0$ in Definition 4.1.4, and obtain a number $\delta > 0$ such that if $0 < |x - c| < \delta$ and $x \in A$, then $|f(x) - L| < \frac{1}{2}L$. Therefore (why?) it follows that if $x \in A \cap V_\delta(c)$, $x \neq c$, then $f(x) > \frac{1}{2}L > 0$.

If $L < 0$, a similar argument applies. Q.E.D.

One-Sided Limits

There are times when a function f may not possess a limit at a point c, yet a limit does exist when the function is restricted to an interval on one side of the cluster point c.

For example, the signum function considered in Example 4.1.10(b), and illustrated in Figure 4.1.2, has no limit at $c = 0$. However, if we restrict the signum function to the interval $(0, \infty)$, the resulting function has a limit of 1 at $c = 0$. Similarly, if we restrict the signum function to the interval $(-\infty, 0)$, the resulting function has a limit of -1 at $c = 0$. These are elementary examples of right-hand and left-hand limits at $c = 0$.

4.3.1 Definition Let $A \in \mathbb{R}$ and let $f : A \to \mathbb{R}$.

(i) If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (c, \infty) = \{x \in A: x > c\}$, then we say that $L \in \mathbb{R}$ is a **right-hand limit of** f **at** c and we write

$$\lim_{x \to c+} f = L \quad \text{or} \quad \lim_{x \to c+} f(x) = L$$

if given any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for all $x \in A$ with $0 < x - c < \delta$, then $|f(x) - L| < \varepsilon$.

(ii) If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (-\infty, c) = \{x \in A: x < c\}$, then we say that $L \in \mathbb{R}$ is a **left-hand limit of** f **at** c and we write

$$\lim_{x \to c-} f = L \quad \text{or} \quad \lim_{x \to c-} f(x) = L$$

if given any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in A$ with $0 < c - x < \delta$, then $|f(x) - L| < \varepsilon$.

4.3.2 Theorem *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of $A \cap (c, \infty)$. Then the following statements are equivalent:*

(i) $\lim_{x \to c+} f = L$.

(ii) *For every sequence (x_n) that converges to c such that $x_n \in A$ and $x_n > c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to L.*

Notes (1) The limits $\lim_{x \to c+} f$ and $\lim_{x \to c-} f$ are called **one-sided limits of** f **at** c. It is possible that neither one-sided limit may exist. Also, one of them may exist without the other existing. Similarly, as is the case for $f(x) := \operatorname{sgn}(x)$ at $c = 0$, they may both exist and be different.

(2) If A is an interval with left endpoint c, then it is readily seen that $f : A \to \mathbb{R}$ has a limit at c if and only if it has a right-hand limit at c. Moreover, in this case the limit $\lim_{x \to c} f$ and the right-hand limit $\lim_{x \to c+} f$ are equal. (A similar situation occurs for the left-hand limit when A is an interval with right endpoint c.)

The reader can show that f can have only one right-hand (respectively, left-hand) limit at a point. There are results analogous to those established in Sections 4.1 and 4.2 for two-sided limits. In particular, the existence of one-sided limits can be reduced to sequential considerations.

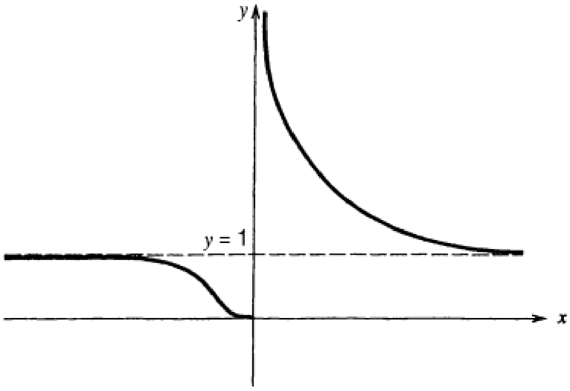

Figure 4.3.1 Graph of $g(x) = e^{1/x}$ $(x \neq 0)$

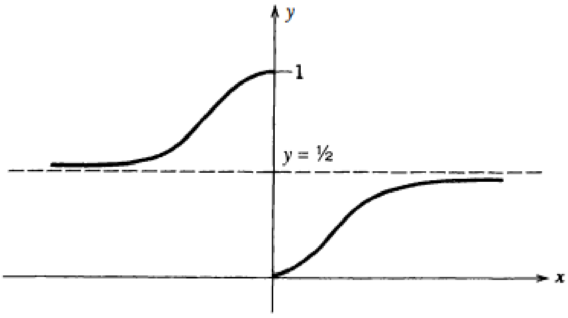

Figure 4.3.2 Graph of $h(x) = 1/(e^{1/x} + 1)$ $(x \neq 0)$

We leave the proof of this result (and the formulation and proof of the analogous result for left-hand limits) to the reader. We will not take the space to write out the formulations of the one-sided version of the other results in Sections 4.1 and 4.2.

The following result relates the notion of the limit of a function to one-sided limits. We leave its proof as an exercise.

4.3.3 Theorem *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of both of the sets $A \cap (c, \infty)$ and $A \cap (-\infty, c)$. Then $\lim_{x \to c} f = L$ if and only if $\lim_{x \to c+} f = L = \lim_{x \to c-} f$.*

4.3.4 Examples **(a)** Let $f(x) := \text{sgn}(x)$.

We have seen in Example 4.1.10(b) that sgn does not have a limit at 0. It is clear that $\lim_{x \to 0+} \text{sgn}(x) = +1$ and that $\lim_{x \to 0-} \text{sgn}(x) = -1$. Since these one-sided limits are different, it also follows from Theorem 4.3.3 that sgn(x) does not have a limit at 0.

(b) Let $g(x) := e^{1/x}$ for $x \neq 0$. (See Figure 4.3.1.)

We first show that g does not have a finite right-hand limit at $c = 0$ since it is not bounded on any right-hand neighborhood $(0, \delta)$ of 0. We shall make use of the inequality

$$(1) \qquad 0 < t < e^t \quad \text{for} \quad t > 0,$$

which will be proved later (see Corollary 8.3.3). It follows from (1) that if $x > 0$, then $0 < 1/x < e^{1/x}$. Hence, if we take $x_n = 1/n$, then $g(x_n) > n$ for all $n \in \mathbb{N}$. Therefore $\lim_{x \to 0+} e^{1/x}$ does not exist in $\mathbb{R}$.

However, $\lim_{x \to 0-} e^{1/x} = 0$. Indeed, if $x < 0$ and we take $t = -1/x$ in (1) we obtain $0 < -1/x < e^{-1/x}$. Since $x < 0$, this implies that $0 < e^{1/x} < -x$ for all $x < 0$. It follows from this inequality that $\lim_{x \to 0-} e^{1/x} = 0$.

(c) Let $h(x) := 1/(e^{1/x} + 1)$ for $x \neq 0$. (See Figure 4.3.2.)

We have seen in part (b) that $0 < 1/x < e^{1/x}$ for $x > 0$, whence

$$0 < \frac{1}{e^{1/x} + 1} < \frac{1}{e^{1/x}} < x,$$

which implies that $\lim_{x \to 0+} h = 0$.

Since we have seen in part (b) that $\lim_{x \to 0-} e^{1/x} = 0$, it follows from the analogue of Theorem 4.2.4(b) for left-hand limits that

$$\lim_{x \to 0-} \left(\frac{1}{e^{1/x} + 1} \right) = \frac{1}{\lim_{x \to 0-} e^{1/x} + 1} = \frac{1}{0 + 1} = 1.$$

Note that for this function, both one-sided limits exist in $\mathbb{R}$, but they are unequal. □

Infinite Limits

The function $f(x) := 1/x^2$ for $x \neq 0$ (see Figure 4.3.3) is not bounded on a neighborhood of 0, so it cannot have a limit in the sense of Definition 4.1.4. While the symbols $\infty (= +\infty)$ and $-\infty$ do not represent real numbers, it is sometimes useful to be able to say that "$f(x) = 1/x^2$ tends to ∞ as $x \to 0$." This use of $\pm\infty$ will not cause any difficulties, provided we exercise caution and *never* interpret ∞ or $-\infty$ as being real numbers.

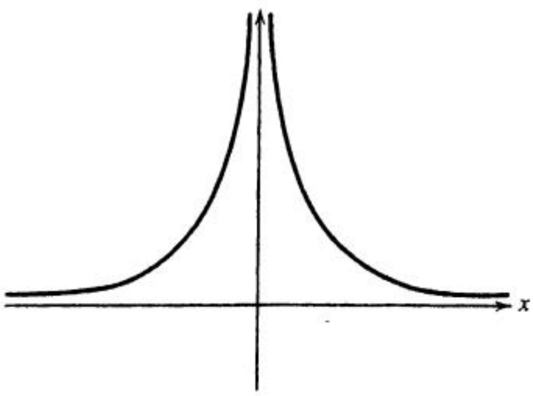

Figure 4.3.3 Graph of $f(x) = 1/x^2 \quad (x \neq 0)$

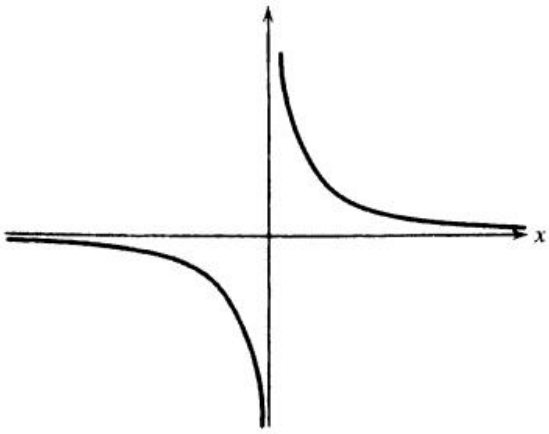

Figure 4.3.4 Graph of $g(x) = 1/x \quad (x \neq 0)$

4.3.5 Definition Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A.

(i) We say that f **tends to** ∞ **as** $x \to c$, and write

$$\lim_{x \to c} f = \infty,$$

if for every $\alpha \in \mathbb{R}$ there exists $\delta = \delta(\alpha) > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta$, then $f(x) > \alpha$.

(ii) We say that f **tends to** $-\infty$ **as** $x \to c$, and write

$$\lim_{x \to c} f = -\infty,$$

if for every $\beta \in \mathbb{R}$ there exists $\delta = \delta(\beta) > 0$ such that for all $x \in A$ with $0 < |x - c| < \delta$, then $f(x) < \beta$.

4.3.6 Examples **(a)** $\lim_{x \to 0} (1/x^2) = \infty$.

For, if $\alpha > 0$ is given, let $\delta := 1/\sqrt{\alpha}$. It follows that if $0 < |x| < \delta$, then $x^2 < 1/\alpha$ so that $1/x^2 > \alpha$.

(b) Let $g(x) := 1/x$ for $x \neq 0$. (See Figure 4.3.4.)

The function g does *not* tend to either ∞ or $-\infty$ as $x \to 0$. For, if $\alpha > 0$ then $g(x) < \alpha$ for all $x < 0$, so that g does not tend to ∞ as $x \to 0$. Similarly, if $\beta < 0$ then $g(x) > \beta$ for all $x > 0$, so that g does not tend to $-\infty$ as $x \to 0$. □

4.3.7 Theorem *Let $A \subseteq \mathbb{R}$, let $f, g : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. Suppose that $f(x) \le g(x)$ for all $x \in A$, $x \neq c$.*

(a) *If $\lim_{x \to c} f = \infty$, then $\lim_{x \to c} g = \infty$.*

(b) *If $\lim_{x \to c} g = -\infty$, then $\lim_{x \to c} f = -\infty$.*

Proof. (a) If $\lim_{x \to c} f = \infty$ and $\alpha \in \mathbb{R}$ is given, then there exists $\delta(\alpha) > 0$ such that if $0 < |x - c| < \delta(\alpha)$ and $x \in A$, then $f(x) > a$. But since $f(x) \le g(x)$ for all $x \in A$, $x \neq c$, it follows that if $0 < |x - c| < \delta(\alpha)$ and $x \in A$, then $g(x) > \alpha$. Therefore $\lim_{x \to c} g = \infty$.

The proof of (b) is similar. Q.E.D.

The function $g(x) = 1/x$ considered in Example 4.3.6(b) suggests that it might be useful to consider one-sided infinite limits. We will define only right-hand infinite limits.

4.3.8 Definition Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (c, \infty) = \{x \in A : x > c\}$, then we say that f **tends to** ∞ [respectively, $-\infty$] as $x \to c+$, and we write

$$\lim_{x \to c+} f = \infty \quad \left[\text{respectively, } \lim_{x \to c+} f = -\infty\right],$$

if for every $\alpha \in \mathbb{R}$ there is $\delta = \delta(\alpha) > 0$ such that for all $x \in A$ with $0 < x - c < \delta$, then $f(x) > \alpha$ [respectively, $f(x) < \alpha$].

Limits at Infinity

It is also desirable to define the notion of the limit of a function as $x \to \infty$. The definition as $x \to -\infty$ is similar.

4.3.10 Definition Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. We say that $L \in \mathbb{R}$ is a **limit of f as $x \to \infty$**, and write

$$\lim_{x\to\infty} f = L \quad \text{or} \quad \lim_{x\to\infty} f(x) = L,$$

if given any $\varepsilon > 0$ there exists $K = K(\varepsilon) > a$ such that for any $x > K$, then $|f(x) - L| < \varepsilon$.

The reader should note the close resemblance between 4.3.10 and the definition of a limit of a sequence.

We leave it to the reader to show that the limits of f as $x \to \pm\infty$ are unique whenever they exist. We also have sequential criteria for these limits; we shall only state the criterion as $x \to \infty$. This uses the notion of the limit of a properly divergent sequence (see Definition 3.6.1).

4.3.11 Theorem *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. Then the following statements are equivalent:*

(i) $L = \lim_{x\to\infty} f$.
(ii) *For every sequence (x_n) in $A \cap (a, \infty)$ such that $\lim(x_n) = \infty$, the sequence $(f(x_n))$ converges to L.*

We leave it to the reader to prove this theorem and to formulate and prove the companion result concerning the limit as $x \to -\infty$.

4.3.12 Examples **(a)** Let $g(x) := 1/x$ for $x \neq 0$.

It is an elementary exercise to show that $\lim_{x\to\infty}(1/x) = 0 = \lim_{x\to-\infty}(1/x)$. (See Figure 4.3.4.)

(b) Let $f(x) := 1/x^2$ for $x \neq 0$.

The reader may show that $\lim_{x\to\infty}(1/x^2) = 0 = \lim_{x\to-\infty}(1/x^2)$. (See Figure 4.3.3.) One way to do this is to show that if $x \geq 1$ then $0 \leq 1/x^2 \leq 1/x$. In view of part (a), this implies that $\lim_{x\to\infty}(1/x^2) = 0$. □

4.3.13 Definition Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Suppose that $(a, \infty) \subseteq A$ for some $a \in A$. We say that f **tends to** ∞ [respectively, $-\infty$] **as** $x \to \infty$, and write

$$\lim_{x\to\infty} f = \infty \quad \left[\text{respectively, } \lim_{x\to\infty} f = -\infty\right]$$

if given any $\alpha \in \mathbb{R}$ there exists $K = K(\alpha) > a$ such that for any $x > K$, then $f(x) > \alpha$ [respectively, $f(x) < \alpha$].

As before there is a sequential criterion for this limit.

4.3.14 Theorem *Let $A \in \mathbb{R}$, let $f : A \to \mathbb{R}$, and suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. Then the following statements are equivalent:*

(i) $\lim_{x\to\infty} f = \infty$ [*respectively*, $\lim_{x\to\infty} f = -\infty$].

(ii) *For every sequence (x_n) in (a, ∞)such that $\lim(x_n) = \infty$, then $\lim(f(x_n)) = \infty$ [respectively, $\lim(f(x_n)) = -\infty$].*

The next result is an analogue of Theorem 3.6.5.

4.3.15 Theorem *Let $A \subseteq \mathbb{R}$, let $f, g : A \to \mathbb{R}$, and suppose that $(a, \infty) \subseteq A$ for some $a \in \mathbb{R}$. Suppose further that $g(x) > 0$ for all $x > a$ and that for some $L \in \mathbb{R}$, $L \neq 0$, we have*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L.$$

(i) *If $L > 0$, then $\lim_{x\to\infty} f = \infty$ if and only if $\lim_{x\to\infty} g = \infty$.*

(ii) *If $L < 0$, then $\lim_{x\to\infty} f = -\infty$ if and only if $\lim_{x\to\infty} g = \infty$.*

Proof. (i) Since $L > 0$, the hypothesis implies that there exists $a_1 > a$ such that

$$0 < \frac{1}{2}L \leq \frac{f(x)}{g(x)} < \frac{3}{2}L \qquad \text{for} \qquad x > a_1.$$

Therefore we have $\left(\frac{1}{2}L\right)g(x) < f(x) < \left(\frac{3}{2}L\right)g(x)$ for all $x > a_1$, from which the conclusion follows readily.

The proof of (ii) is similar. Q.E.D.