APPLICATIONS OF LAPLACE TRANSFORMS

PART I : Applications of Laplace transforms to ordinary differential equations

3.1. Solution of ordinary differential equation with constant coefficients.

Suppose we wish to solve the nth order ordinary differential equation with constant coefficients

$$d^n y/dt^n + a_1 (d^{n-1} y/d t^{n-1}) + a_2 (d^{n-2} y/d t^{n-2}) + \ldots + a_n y = F(t), \quad \ldots(1)$$

where $a_1, a_2, \ldots, a_n$ are constants, subject to the initial conditions

$$y(0) = k_0, y'(0) = k_1, y''(0) = k_2, \ldots, y^{(n-1)}(0) = k_{n-1}, \quad \ldots(2)$$

where $k_0, k_1, \ldots, k_{n-1}$ are constants.

On taking the Laplace transform of both sides of (1) and using (2), we obtain an algebraic equation (which is known as "subsidiary equation") for determination of L $\{y(t)\}$.The required solution is then obtained by finding the inverse Laplace transform of L $\{y(t)\}$. Since the formulas of Art 1.15, Chapter 1 depend on the initial conditions of the differential equation, these conditions are automatically obtained in the final solution of the given differential equation when the inverse is found.

Very important note. Students are adviced to commit to memory all rules of finding Laplace transforms and inverse Laplace transforms discussed in chapters 1 and 2.

Notations. In what follows, we shall adopt the following notations:

(*i*) $dy/dt = Dy = y'(t) = y^{(1)}(t), d^2y/dt^2 = D^2 y = y''(t) = y^{(2)}(t), \ldots$
$d^n y/dt^n = D^n y = y^{(n)}(t)$ etc.

(*ii*) At $t = 0$, we have

$$y(0) = y_0, y'(0) = y_1, y''(0) = y_2, \ldots y^{(n)}(0) = y_n.$$

In some problems, the dependent variable may be x or z etc in place of y in (1). Then we make the corresponding changes in the notations which have just been discussed.

The whole procedure of solution will become clear by reading the following solved examples.

Ex.1. *Solve* $(D^2 - 1)\, y = a \cosh nt$, *if* $y = Dy = 0$ *when* $t = 0$.

Sol. Re-writing the given equation and conditions, we have

$$y'' - y = a \cosh nt, \qquad \dots (1)$$

with initial conditions: $\quad y(0) = 0$ and $y'(0) = 0. \qquad \dots (2)$

Taking Laplace transform of both sides of (1), we get

$$L\{y''\} - L\{y\} = L\{a \cosh nt\}$$

or $\quad s^2 L\{y\} - s\,y(0) - y'(0) - L\{y\} = as/(s^2 - n^2)$

or $\quad (s^2 - 1) L\{y\} - s.0 - 0 = as/(s^2 - n^2)$, using (2)

or $$L\{y\} = \frac{as}{(s^2-1)(s^2-n^2)} = \frac{as}{n^2-1}\left\{\frac{(s^2-1)-(s^2-n^2)}{(s^2-1)(s^2-n^2)}\right\} \qquad \textbf{\textit{(Note)}}$$

(on rewriting for sake of partial fractions)

or $$L\{y\} = \frac{as}{n^2-1}\left[\frac{1}{s^2-n^2} - \frac{1}{s^2-1}\right] = \frac{a}{n^2-1}\left[\frac{s}{s^2-n^2} - \frac{s}{s^2-1}\right]$$

Taking inverse Laplace transform of both sides, we have

$$y = \frac{a}{n^2-1}\left[L^{-1}\left\{\frac{s}{s^2-n^2}\right\} - L^{-1}\left\{\frac{s}{s^2-1}\right\}\right] = \frac{a}{n^2-1}\,(\cosh nt - \cosh t).$$

Ex.2. *Solve* $(D^2 + 1)y = 6 \cos 2t$, *if* $y = 3$, $Dy = 1$, *when* $t = 0$.

(*Meerut 91, 93, Purvanchal 92, 94, 96*)

Sol. Re-writing the given equation and conditions, we have

$$y'' + y = 6 \cos 2t, \qquad \dots (1)$$

with initial conditions: $\quad y(0) = 3$, and $y'(0) = 1. \qquad \dots (2)$

Taking Laplace transform of both sides of (1), we get

$$L\{y''\} + L\{y\} = 6\, L\{\cos 2t\}$$

or $\quad s^2 L\{y\} - s\,y(0) - y'(0) + L\{y\} = 6s/(s^2 + 2^2)$

or $\quad (s^2 + 1) L\{y\} - 3s - 1 = 6s/(s^2 + 4)$, using (2)

or $$L\{y\} = \frac{3s}{s^2+1} + \frac{1}{s^2+1} + \frac{6s}{(s^2+1)(s^2+4)}$$

or $$L\{y\} = \frac{3s}{s^2+1} + \frac{1}{s^2+1} + 2s\left\{\frac{1}{s^2+1} - \frac{1}{s^2+4}\right\}$$

$$\therefore \quad L\{y\} = 5\cdot\frac{s}{s^2+1} + \frac{1}{s^2+1} - 2\cdot\frac{s}{s^2+2^2}. \qquad \dots (3)$$

Taking inverse Laplace transform of both sides of (3), we get

$$y = 5 \cos t + \sin t - 2 \cos 2t.$$

... the required solution.

Ex. 5 (b). *Using Laplace transform, determine the solution of* $(d^2y/dt^2) + 3\,(dy/dt) + 2y = e^{-t}$, $y\,(0) = y'\,(0) = 0.$ **(Lucknow 1998)**

Sol. Re-writing the given equation and conditions, we have

$$y'' + 3y' + 2y = e^{-t} \qquad \text{... (1)}$$

with the initial conditions: $\quad y\,(0) = 0$ and $y'\,(0) = 0.$ ... (3)

Taking Laplace transform of both sides of (1), we have

$$\mathrm{L}\,\{y''\} + 3\,\mathrm{L}\,\{y'\} + 2\,\mathrm{L}\,\{y\} = \mathrm{L}\,\{e^{-t}\}$$

or $\quad s^2\,\mathrm{L}\,\{y\} - sy\,(0) - y'\,(0) + 3\,\{s\,\mathrm{L}\,\{y\} - y\,(0)\} + 2\,\mathrm{L}\,\{y\} = 1/(s+1)$

or $\quad (s^2 + 3s + 2)\,\mathrm{L}\,\{y\} = 1/(s+1)$, using (2)

or $\quad \mathrm{L}\,\{y\} = \dfrac{1}{(s^2+3s+2)\,(s+1)} = \dfrac{1}{(s+2)\,(s+1)^2} = \dfrac{1}{s+2} - \dfrac{1}{s+1} + \dfrac{1}{(s+1)^2}$

[on resolving into partial fractions]

[Taking inverse transform of both sides of (3), we get

$$y = \mathrm{L}^{-1}\left\{\frac{1}{s+2}\right\} - \mathrm{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathrm{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = e^{-2t} - e^{-t} - e^{-t}\,\mathrm{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

[using first shifting theorem in the last term]

or $\quad y = e^{-2t} - e^{-t} - e^{-t}\cdot t.$ $\qquad [\because \mathrm{L}^{-1}\{1/s^{n+1}\} = t^n/n!.]$

Ex. 6. *Solve* $(D^2 + D)\,x = 2$, *when* $x\,(0) = 3$, $x'\,(0) = 1.$

(Meerut 91, Kanpur 94, 96)

Sol. Re-writing the given equation and conditions, we have

$$x'' + x' = 2, \qquad \text{... (1)}$$

with initial conditions: $\quad x\,(0) = 3$ and $x'\,(0) = 1.$ ... (2)

Taking Laplace transform of both sides of (1), we have

$$\mathrm{L}\,\{x''\} + \mathrm{L}\,\{x'\} = 2\,\mathrm{L}\,\{1\}$$

or $\quad s^2\,\mathrm{L}\,\{x\} - s\,x\,(0) - x'\,(0) + s\,\mathrm{L}\,\{x\} - x\,(0) = 2\cdot(1/s)$, using (2)

or $\quad (s^2 + s)\,\mathrm{L}\,\{x\} - 3s - 1 - 3 = 2/s \quad$ or $\quad (s^2 + s)\,\mathrm{L}\,\{x\} = 3s + 4 + 2/s$

or $\quad \mathrm{L}\,\{x\} = \dfrac{3s^2+4s+2}{s^2(s+1)} = \dfrac{s^2+2(s^2+2s+1)}{s^2(s+1)} = \dfrac{1}{s+1} + \dfrac{2}{s} + \dfrac{2}{s^2}$ ... (3)

Taking inverse Laplace transform of both sides of (3), we get

$$x = e^{-t} + 2 + 2t$$

Ex. 7. *Solve* $(D^2 - 2D + 2)\,y = 0$, $y = Dy = 1$, *when* $t = 0.$

Ex. 8. *Solve* $(D^2 + 2D + 1)\,y = 3t\,e^{-t}$, *subject to the conditions,* $y = 4$, $Dy = 2$ *when* $t = 0.$

Ex.10. *Solve* $(D^2 + 2D + 5)\, y = e^{-t} \sin t$, $y(0) = 0$, $y'(0) = 1$.

(Delhi 97, Manglore 97)

Sol. Re-writing the given equation and conditions, we have

$$y'' + 2y' + 5y = e^{-t} \sin t, \qquad \text{... (1)}$$

with initial conditions: $y(0) = 0$ and $y'(0) = 1$. ... (2)

Taking Laplace transform of both sides of both sides of (1), we get

$$L\{y''\} + 2\,L\{y'\} + 5\{y\} = L\{e^{-t} \sin t\}$$

or $s^2 L\{y\} - s\,y(0) - y'(0) + 2\,[s\,L\{y\} - y(0)] + 5\,L\{y\} = f(s+1)$,

[using first shifting theorem 3.11 and taking $f(s) = L\{\sin t\} = 1/(s^2+1)$.]

or $(s^2 + 2s + 5)\, L\{y\} - 0 - 1 - (2 \times 0) = 1/[(s+1)^2 + 1]$, by (2)

or $$L\{y\} = \frac{1}{(s^2+2s+5)} + \frac{1}{(s^2+2s+5)\,[(s+1)^2+1]}$$

or $$y = L^{-1}\left\{\frac{1}{(s+1)^2+4} + \frac{1}{\{(s+1)^2+4\}\,\{(s+1)^2+1\}}\right\}$$

$$= e^{-t}\, L^{-1}\left\{\frac{1}{s^2+4} + \frac{1}{(s^2+4)\,(s^2+1)}\right\} = e^{-t}\, L^{-1}\left\{\frac{1}{s^2+4} + \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right)\right\}$$

$$= e^{-t}\, L^{-1}\left\{\frac{1}{3}\,\frac{1}{s^2+1} + \frac{2}{3}\,\frac{1}{s^2+4}\right\} = e^{-t}\left[\frac{1}{3}\sin t + \frac{2}{3}\cdot\frac{1}{2}\sin 2t\right]$$

$\therefore$ $y = (1/3)\, e^{-t} (\sin t + \sin 2t)$ is required solution.

Ex.11. *Solve* $(D^2 + 1)\, y = \sin t \cos 2t$, $t > 0$, *if* $y = 1$, $Dy = 0$ when $t = 0$

Sol. Re-writing the given equation and conditions, we get

$$y'' + y = (1/2)\,(\cos t - \cos 3t), \qquad \text{... (1)}$$

with initial conditions: $y(0) = 1$ and $y'(0) = 0$. ... (2)

Taking Laplace transform of both sides of (1), we get

$$L\{y''\} + L\{y\} = (1/2)\,[L\{\cos t\} - L\{\cos 3t\}]$$

or $s^2 L\{y\} - s\,y(0) - y'(0) + L(y) = (1/2)\,\{s/(s^2+1) - s/(s^2+9)\}$

or $(s^2+1)\, L\{y\} - s - 0 = s/2(s^2+1) - s/2(s^2+9)$, using (2)

or $$L\{y\} = \frac{s}{s^2+1} + \frac{1}{2}\,\frac{s}{(s^2+1)^2} - \frac{1}{2}\,\frac{s}{(s^2+1)\,(s^2+9)}$$

$$= \frac{s}{s^2+1} + \frac{1}{2}\,\frac{s}{(s^2+1)^2} - \frac{s}{16}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$$

$\therefore$ $$L\{y\} = \frac{15}{16}\,\frac{s}{s^2+1} + \frac{1}{2}\,\frac{s}{(s^2+1)^2} - \frac{1}{16}\,\frac{s}{s^2+9} \qquad \text{... (3)}$$

Taking inverse Laplace transform of both sides of (3), we get

$$y = \frac{15}{16} L^{-1}\left\{\frac{s}{s^2+1^2}\right\} + \frac{1}{2} L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} - \frac{1}{16} L^{-1}\left\{\frac{s}{s^2+3^2}\right\}$$

or $$y = \frac{15}{10}\cos t - \frac{1}{2}\cdot\frac{1}{2} L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\} - \frac{1}{16}\cos 3t \qquad \textbf{(Note)}$$

$$= (15/16)\cos t - (1/4)(-1)^1\, t\, L^{-1}\{1/(s^2+1)\} - (1/16)\cos 3t$$

Ex.16. *Solve $(D^2 - D - 6)\, y = 2$, $t > 0$ if $y = 1$, $Dy = 0$ when $t = 0$.* ***(Meerut 86)***

Sol. Re-writing the given equation and conditions, we get

$$y'' - y' - 6y = 2 \qquad \text{... (1)}$$

with initial conditions: $y(0) = 1$ and $y'(0) = 0$. ... (2)

Taking Laplace transform of both sides of (1), we get

$$L\{y''\} - L\{y'\} - 6\,L\{y\} = 2\,L\{1\}$$

or $s^2 L\{y\} - s\,y(0) - y'(0) - [s\,L\{y\} - y(0)] - 6\,L\{y\} = 2/s$

or $(s^2 - s - 6)\,L\{y\} - s - 0 + 1 = 2/s$, using (2)

or $$L\{y\} = \frac{s^2-s+2}{s(s^2-s-6)} = \frac{s^2-s+2}{s(s+2)(s-3)}$$

$\therefore$ $y = L^{-1}\{(s^2 - s + 2)/s(s+2)(s-3)\}$. ... (3)

Let $$\frac{s^2-s+2}{s(s+2)(s-3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-3} \qquad \text{... (4)}$$

Multiply both sides of (4) by s and let $s \to 0$, then

$$A = \lim_{s\to 0} \frac{s^2-s+2}{(s+2)(s-3)} = \frac{2}{(2)(-3)} = -\frac{1}{3}.$$

Multiply both sides of (4) by $(s + 2)$ and let $s \to -2$, then

$$B = \lim_{s\to -2} \frac{s^2-s+2}{s(s-3)} = \frac{(-2)^2+2+2}{(-2)(-5)} = \frac{4}{5}.$$

Multiply both sides of (4) by $(s - 3)$ and let $s \to 3$, then

$$C = \lim_{s\to 3} \frac{s^2-s+2}{s(s+2)} = \frac{3^2-3+2}{(3)(5)} = \frac{8}{15}.$$

$$\therefore \quad (4) \Rightarrow \frac{s^2-s+2}{s(s+2)(s-3)} = -\frac{1}{3s} + \frac{4}{5(s+2)} + \frac{8}{15(s-3)}$$

Then $$(3) \Rightarrow y = -\frac{1}{3} L^{-1}\left\{\frac{1}{s}\right\} + \frac{4}{5} L^{-1}\left\{\frac{1}{s+2}\right\} + \frac{8}{15} L^{-1}\left\{\frac{1}{s-3}\right\}$$

or $$y = -(1/3) + (4/5)\,e^{-2t} + (8/15)\,e^{3t}.$$

Ex.20. *Solve $(D^2 + 6D + 9)\, y = \sin t$, where $y(0) = 1$, $y'(0) = 0$.*

(Meerut 96, Rohilkhand 88, 89)

Sol. Given that $$y'' + 6y' + 9y = \sin t \qquad \text{... (1)}$$

with initial conditions: $$y(0) = 1 \text{ and } y'(0) = 0. \qquad \text{... (2)}$$

Taking Laplace transform of both sides of (1), we get

$$L\{y''\} + 6L\{y'\} + 9L\{y\} = L\{\sin t\}$$

or $$s^2 L\{y\} - sy(0) - y'(0) + 6[sL\{y\} - y(0)] + 9L\{y\} = 1/(s^2+1)$$

or $$(s^2 + 6s + 9)L\{y\} - s - 6 = 1/(s^2+1), \text{ using (2)}$$

or $$(s+3)^2 L\{y\} = s + 6 + 1/(s^2+1) = (s+3) + 3 + 1/(s^2+1)$$

or $$L\{y\} = \frac{1}{(s+3)} + \frac{3}{(s+3)^2} + \frac{1}{(s+3)^2(s^2+1)}. \qquad \text{... (3)}$$

Let $$\frac{1}{(s+3)^2(s^2+1)} = \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{Cs+D}{s^2+1}. \qquad \text{... (4)}$$

Multiply both sides of (4) by $(s+3)^2$ and let $s \to -3$, then

$$B = \lim_{s\to -3} 1/(s^2+1) = 1/(9+1) = 1/10.$$

Then, (4) $\Rightarrow$ $$\frac{1}{(s+3)^2(s^2+1)} = \frac{A}{s+3} + \frac{1}{10(s+3)^2} + \frac{Cs+D}{s^2+1}. \qquad \text{... (5)}$$

Multiplying both sides of (5) by $(s+3)^2(s^2+1)$, we get

$$1 = A(s+3)(s^2+1) + (1/10)(s^2+1) + (Cs+D)(s+3)^2$$

or $$1 = A(s+3)(s^2+1) + (1/10)(s^2+1) + (Cs+D)(s^2+6s+9)$$

or $$1 = 3A + (1/10) + 9D + s(A + 9C + 6D) + s^2[3A + (1/10) + D + 6C] + s^3(A+C). \qquad \text{... (6)}$$

Equating the coefficients of s^3, s and constant terms on both sides of the identity (6), $A + C = 0$, $A + 9C + 6D = 0$ and $3A + (1/10) + 9D = 1$.

Solving these, $\qquad A = 3/50,\ C = -3/50,\ D = 4/50.$

$\therefore$ (5) $\Rightarrow$ $$\frac{1}{(s+3)^2(s^2+1)} = \frac{3}{50(s+3)} + \frac{1}{10(s+3)^2} - \frac{3s-4}{50(s^2+1)}.$$

$\therefore$ (3) $\Rightarrow$ $$L\{y\} = \frac{1}{s+3} + \frac{3}{(s+3)^2} + \frac{3}{50(s+3)} + \frac{1}{10(s+3)^2} - \frac{3s-4}{50(s^2+1)}.$$

or $$L\{y\} = \frac{53}{50(s+3)} + \frac{31}{10(s+3)^2} - \frac{3s}{50(s^2+1)} + \frac{2}{25(s^2+1)}.$$

or $$y = \frac{53}{50}e^{-3t} + \frac{31}{10}e^{-3t} L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{3}{50}\cos t + \frac{2}{25}\sin t$$

[Using first shifting theorem]

or $$y = (53/50)\,e^{-3t} + (31/10)\,t\,e^{-3t} - (3/50)\cos t + (2/25)\sin t$$

Solve the following differential equations:

1. $(D+1)\,y = 0,\ t > 0$, given that $y = y_0$, when $t = 0$.
2. $(D+1)\,y = 1$, given that $y = 2$ when $t = 0$.
3. $(D^2+1)\,y = 0$, if $y = 1$, $dy/dt = 0$, when $t = 0$. **(*Meerut 85*)**
4. $(D^2+9)\,y = 18t$, if $y(0) = 0$, $y(\pi/2) = 0$. **(*Osmania 2004*)**
5. $(D^2+3D+2)\,y = 0$, $y = y_0$ and $Dy = y_1$ at $t = 0$. **(*Meerut 91, Rohilkhand 94*)**
6. $(D^2+6D+25)\,y = 208\,e^{3t},\ t > 0$, if $y = 1$, $Dy = 0$ when $t = 0$.
7. $(D^3+1)\,y = 1,\ t > 0$ if $y = Dy = D^2y = 0$ when $t = 0$.
8. $(D^3+D)\,y = e^{2t}$, $y(0) = y'(0) = y''(0) = 0$.
9. $(D^3-D)\,y = 2\cos t$, $y = 3$, $Dy = 2$, $D^2y = 1$ when $t = 0$.
10. $(D^2+4D+4)\,x = \sin wt,\ t > 0$ with x_0 and x_1 for values of x and Dx when $t = 0$.

3.2. Solution of ordinary differential equations with variable coefficients

Suppose the given differential equation contain a term of the form

$t^m\,y^{(n)}(t)$ *i.e.*, $t^m \dfrac{d^n y(t)}{dt^n}$ the Laplace transform of which is

$$(-1)^m \frac{d^m}{ds^m} L\{y^{(n)}(t)\}.$$

Ex.1. *Solve* $[t\,D^2 + (1-2t)\,D - 2]\,y = 0,\ y(0) = 1,\ y'(0) = 2.$

Sol. Given $\qquad t\,y'' + y' - 2t\,y' - 2y = 0, \qquad$ ... (1)

with initial conditions: $\qquad y(0) = 1$ and $y'(0) = 2. \qquad$ ... (2)

Taking Laplace transform of both sides of (1), we have

$$L\{t\,y''\} + L\{y'\} - 2L\{ty'\} - 2L\{y\} = 0$$

or $$(-1)^1 \frac{d}{ds} L\{y''\} + L\{y'\} - 2(-1)^1 \frac{d}{ds} L\{y'\} - 2L\{y\} = 0$$

or $$-\frac{d}{ds}[s^2 L\{y\} - sy(0) - y'(0)] + sL\{y\} - y(0) + 2\frac{d}{ds}[sL\{y\} - y(0)] - 2L\{y\} = 0$$

or $$-\frac{d}{ds}[s^2\bar{y} - s - 2] + s\bar{y} - 1 + 2\frac{d}{ds}[s\bar{y} - 1] - 2\bar{y} = 0, \text{ by (2)} \qquad ...(3)$$

where we write $\quad L\{y\} = \bar{y}(s)$. ... (4)

From (3), $-\left[s^2 \dfrac{d\bar{y}}{ds} + 2s\bar{y} - 1\right] + s\bar{y} - 1 + 2\left[s\dfrac{d\bar{y}}{ds} + \bar{y}\right] - 2\bar{y} = 0$

or $\quad (s^2 - 2s)\dfrac{d\bar{y}}{ds} = s\bar{y} \quad$ or $\quad \dfrac{d\bar{y}}{\bar{y}} + \dfrac{ds}{s-2} = 0.$

Integrating, $\log \bar{y} + \log(s-2) = \log C$ or $\bar{y} = C/(s-2)$.

$\therefore \quad L\{y\} = C/(s-2)$, using (4). ... (5)

Taking inverse transform of both sides of (5), we have

$y(t) = C\,L^{-1}\{1/(s-2)\} = Ce^{2t}$. ... (6)

Putting $t = 0$ in (6), we get $y(0) = C$ so that $C = 1$, using (2). Hence, from (6), $y = e^{2t}$, which is the required solution

Ex.2. *Solve $t y'' + y' + 4ty = 0$, if $y(0) = 3$, $y'(0) = 0$.*

Sol. Given $\quad t y'' + y' + 4ty = 0$ ... (1)

with initial conditions: $y(0) = 3$ and $y'(0) = 0$. ... (2)

Taking Laplace transform of both sides of (1), we get

$$L\{ty''\} + L\{y'\} + 4L\{ty\} = 0$$

or $\quad (-1)^1 \dfrac{d}{ds}L\{y''\} + L\{y'\} + 4(-1)^1 \dfrac{d}{ds}L\{y\} = 0$

or $\quad -\dfrac{d}{ds}\left[s^2 L\{y\} - sy(0) - y'(0)\right] + sL\{y\} - y(0) - 4\dfrac{d}{ds}L\{y\} = 0$

or $\quad -\dfrac{d}{ds}[s^2 L\{y\} - 3s] + sL\{y\} - 3 - 4\dfrac{d}{ds}L\{y\} = 0$, using (2) ... (3)

Let $L\{y\} = \bar{y}(s)$. ... (4)

Then (3) $\Rightarrow -\dfrac{d}{ds}(s^2\bar{y} - 3s) + s\bar{y} - 3 - 4\dfrac{d\bar{y}}{ds} = 0$

or $\quad -\left[s^2\dfrac{d\bar{y}}{ds} + 2s\bar{y} - 3\right] + s\bar{y} - 3 - 4\dfrac{d\bar{y}}{ds} = 0$

or $\quad (s^2 + 4)\dfrac{d\bar{y}}{ds} + s\bar{y} = 0 \quad$ or $\quad \dfrac{d\bar{y}}{\bar{y}} + \dfrac{1}{2}\cdot\dfrac{2s}{s^2+4}ds = 0.$

Integrating, $\log \bar{y} + (1/2)\log(s^2+4) = \log c$ or $\bar{y} = c/\sqrt{s^2+4}$.

$\therefore \quad L\{y\} = c/\sqrt{(s^2+4)}$, using (4). ... (5)

(5) $\Rightarrow y(t) = L^{-1}\left\{\dfrac{c}{\sqrt{(s^2+4)}}\right\} = c\,L^{-1}\left\{\dfrac{1}{\sqrt{(s^2+2^2)}}\right\} = c\,J_0(2t)$ ... (6)

[$\because$ from Ex 1 (*i*), page 51, $L\{J_0(at)\} = 1/\sqrt{(s^2+a^2)}$]

Putting $t = 0$ in (6), we get

$y(0) = C\,J_0(0)$ or $3 = C$, using (2) and noting that $J_0(0) = 1$.

Hence from (6), $y = 3J_0(2t)$, which is the required solution

3.3. Solution of simultaneous ordinary differential equations

Simultaneous ordinary differential equations involve more than one dependent variable and Laplace transform is needed for each variable. The procedure is to solve the simultaneous algebraic equations which appear on taking Laplace transformation and finally invert to recover each dependent variable. We now explain this method in what follows.

Ex.1. *Solve $dx/dt = 2x - 3y$, $dy/dt = y - 2x$, if $x(0) = 8$ and $y(0) = 3$.*

Or Solve $(D-2)x + 3y = 0$, $2x + (D-1)y = 0$ if $x(0) = 8$ and $y(0) = 3$

Sol. Given $\qquad x'(t) - 2x + 3y = 0$

nd $\qquad 2x + y'(t) - y = 0$

ith initial conditions: $\quad x(0) = 8$ and $y(0) = 3$.

Let $L\{x(t)\} = \bar{x}(s)$ and $L\{y(t)\} = \bar{y}(s)$. Taking the Laplace transform of both sides of (1A) and (2A), we get

$$L\{x'(t)\} - 2L\{x(t)\} + 3L\{y(t)\} = 0 \qquad \text{(1B)}$$

and
$$2L\{x(t)\} + L\{y'(t)\} - L\{y(t)\} = 0 \qquad \text{... (2B)}$$

or
$$s\bar{x} - x(0) - 2\bar{x} + 3\bar{y} = 0 \qquad \text{... (1C)}$$

and
$$2\bar{x} + s\bar{y} - y(0) - \bar{y} = 0. \qquad \text{... (2C)}$$

Using (3), (1C) and (2C) reduce to

$$(s-2)\bar{x} + 3\bar{y} - 8 = 0 \qquad \text{... (1D)}$$

and
$$2\bar{x} + (s-1)\bar{y} - 3 = 0. \qquad \text{... (2D)}$$

Solving (1D) and (2D) for $\bar{x}$ and $\bar{y}$ by cross-multiplication, we get

$$\frac{\bar{x}}{-9+8(s-1)} = \frac{\bar{y}}{-16+3(s-2)} = \frac{1}{(s-1)(s-2)-6}$$

or
$$\frac{\bar{x}}{8s-17} = \frac{\bar{y}}{3s-22} = \frac{1}{s^2-3s-4}.$$

Hence
$$\bar{x} = \frac{8s-1}{(s-4)(s+1)} = \frac{3}{s-4} + \frac{5}{s+1}. \qquad \text{... (1E)}$$

and
$$\bar{y} = \frac{3s-22}{(s-4)(s+1)} = \frac{5}{s+1} - \frac{2}{s-4}. \qquad \text{... (2E)}$$

[on resolving into partial fractions]

Taking the inverse Laplace transform of (1E) and (2E), we get

$$x = 3L^{-1}\left\{\frac{1}{s-4}\right\} + 5L^{-1}\left\{\frac{1}{s+1}\right\} = 3e^{4t} + 5e^{-t}$$

Ex.5. *Solve* $(D^2 - 1)\, x + 5Dy = t,\ - 2Dx + (D^2 - 4)\, y = -2,$
if $y = x = 0 = Dx = Dy$ *when* $t = 0$. **(Meerut 86, Rohilkhand 90)**

Ex.7. *Solve* $Dx + Dy = t,\ D^2 x - y = e^{-t}$,
if $x(0) = 3,\ x'(0) = -2,\ y(0) = 0.$ **(Meerut 87, 94)**

Sol. Given $x'(t) + y'(t) = t$ ... (1A)

and $x''(t) - y(t) = e^{-t}$. ... (2A)

with initial conditions: $x(0) = 3,\ x'(0) = -2,\ y(0) = 0.$ ... (3)

Let $L\{x(t)\} = \bar{x}$ and $L\{y(t)\} = \bar{y}$, Taking Laplace the transform of both sides of (1A) and (2A), we get

$L\{x'(t)\} + L\{y'(t)\} = L\{t\}$ ... (1B)

and $L\{x''(t)\} - L\{y(t)\} = L\{e^{-t}\}$ ... (2B)

or $s\bar{x} - x(0) + s\bar{y} - y(0) = 1/s^2$ ... (1C)

and $s^2\bar{x} - sx(0) - x'(0) - \bar{y} = 1/(s+1).$ ... (2C)

Using (3), (1C) and (2C) reduce to

$s\bar{x} + s\bar{y} = 3 + (1/s^2) = (3s^2+1)/s^2$ ... (1D)

and $s^2\bar{x} - \bar{y} = 3s - 2 + 1/(s+1) = (3s^2 + s - 1)/(s+1)$ ... (2D)

Solving (1D) and (2D) for $\bar{x}$ and $\bar{y}$, we get

$$\bar{x} = \frac{3s^2+1}{s^3(s^2+1)} + \frac{3s^2+s-1}{(s+1)(s^2+1)},\quad \bar{y} = \frac{3s^2+1}{s^2(s^2+1)} - \frac{3s^2+s-1}{(s+1)(s^2+1)}$$

Resolving into partial fractions, these give

$$\bar{x} = \frac{2}{s} + \frac{1}{s^3} + \frac{1}{2(s+1)} - \frac{3}{2(s^2+1)} + \frac{s}{2(s^2+1)}$$

and $$\bar{y} = \frac{1}{s} - \frac{1}{2(s+1)} + \frac{3}{2(s^2+1)} - \frac{s}{2(s^2+1)}.$$

Taking inverse Laplace transforms, there give

$x = 2 + (1/2)\, t^2 + (1/2)\, e^{-t} - (3/2)\sin t + (1/2)\cos t$

and $y = 1 - (1/2)\, e^{-t} + (3/2)\sin t - (1/2)\cos t$

Solve the following simultaneous differential equations:

1. $Dx + Dy = t,\ D^2 x - y = e^{-t}$, if $x(0) = y(0) = x'(0) = 0.$
(Meerut 95, Rohilkhand 89)

Ans. $x = (1/2)(t^2 + e^{-t} + \cos t + \sin t) - 1,\ y = 1 - (1/2)(e^{-t} + \cos t + \sin t).$

2. $(dx/dt) + 5x - 2y = t,\ (dy/dt) + 2x + y = 0$, if $x = y = 0$ when $t = 0$
(S.V. University (A.P.) 1997)

3. $(dx/dt) - 6x + 3y = 8e^t,\ (dy/dt) - 2x - y = 4e^t$ if $x(0) = -1$ and $y(0) = 0$
(S.V. University (A.P.) 1997)

4. $\dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} + 3x = 15\, e^{-t},\ \dfrac{d^2y}{dt^2} - 4\dfrac{dx}{dt} + 3y = 15\sin 2t$, if $x(0) = 35$, $x'(0) = -48,\ y(0) = 27$ and $y'(0) = -55.$

Ans. $x = 30\cos t - 15\sin 3t - 3\, e^{-t} + 2\cos 2t,$
$y = 30\cos 3t - 60\sin t - 3\, e^{-t} + \sin 2t.$

5. $Dx + 2D^2y = e^{-t},\ (D + 2)\, x - y = 1$ if $x(0) = y(0) = y'(0) = 0$

Ans. $x = 1 + e^{-t} - e^{-at} - e^{-bt},\ y = 1 + e^{-t} - be^{-at} - ae^{-bt},$

where $a = (2 - \sqrt{2})/2$ and $b = (2 + \sqrt{2})/2.$