II **(15 Marks)**	**Laplace Transform:** Definition of Laplace transform, Existence theorem for Laplace transform. Linearity property of Laplace transform, Laplace transform of some elementary functions. (algebraic functions, trigonometric functions, exponential functions, hyperbolic functions). First Shifting theorem, Second shifting theorem, Change of scale property, Laplace transform of derivatives, Laplace transform of Integrals.	12	04	-	16

Laplace transform. Definition.

Given a function F(t) defined for all real $t \geq 0$, the Laplace transform of F (t) is a function of a new variable s given by

$$L\{F(t), s\} = L\{F(t)\} = f(s) = \bar{F}(s) = \int_0^\infty e^{-st} F(t)\, dt \qquad \dots (1)$$

Sufficient conditions for the existence of Laplace transform

Theorem. *If F (t) is a function of class A, L { F (t)} exists.*

Proof. Since $F(t)$ is of exponential order, say σ, we can find constants $\sigma, m\ (>0)$ and $t_0\ (>0)$ such that

$$|F(t)| < m e^{\sigma t} \text{ for } t \geq t_0. \qquad \dots (1)$$

Now, $L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = \int_0^{t_0} e^{-st} F(t)\, dt + \int_{t_0}^\infty e^{-st} F(t)\, dt$

or $$L\{F(t)\} = I_1 + I_2, \text{ say} \qquad \dots (2)$$

Since F (t) is piecewise continuous on every finite interval $0 \leq t \leq t_0$, I_1 exists. Again, we have

$$|I_2| = \left|\int_{t_0}^\infty e^{-st} F(t)\, dt\right| \leq \int_{t_0}^\infty e^{-st} |F(t)|\, dt < m \int_{t_0}^\infty e^{-st} e^{\sigma t}\, dt, \text{by (1)}$$

$$\Rightarrow \quad |I_2| < m \left[-\frac{e^{-(s-\sigma)t}}{(s-\sigma)}\right]_{t_0}^\infty = \frac{m\, e^{-(s-\sigma)t_0}}{s-\sigma}. \qquad \dots (3)$$

Now, when $s > \sigma$, then $e^{-(s-\sigma)t_0} \to 0$ as $t \to \infty$. Hence (3) shows that $|I_2|$ is finite for all $t_0 > 0$ when $s > \sigma$ and hence I_2 is also convergent. Then from (2), it follows that $L\{F(t)\}$ exists for all $s > \sigma$.

1.8. Linearity property of Laplace transforms. If c_1 and c_2 be constants, then $L\{c_1 F_1(t) + c_2 F_2(t)\} = c_1 L\{F_1(t) + c_2 L\{F_2(t)\}$.

Proof. By definition, we have

$$L\{c_1 F_1(t) + c_2 F_2(t)\} = \int_0^\infty e^{-st}\{c_1 F_1(t) + c_2 F_2(t)\}dt$$

$$= c_1 \int_0^\infty e^{-st} F_1(t)\,dt + c_2 \int_0^\infty e^{-st} F_2(t)\,dt$$

$$= c_1 L\{F_1(t)\} + c_2 L\{F_2(t)\}, \text{ by definition.}$$

1.9. Laplace transforms of some elementary functions

(i) Laplace transform of the function F (t) = 1.

Sol. By definition of Laplace transform, $L\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$.

$$\therefore \quad L\{1\} = \int_0^\infty e^{-st}(1)\,dt = \left[-\frac{e^{-st}}{s}\right]_0^\infty = \frac{1}{s}, \text{ provided } s > 0$$

Since the integral is convergent if $s > 0$ and divergent if $s \le 0$, hence the condition $s > 0$ is necessary for the existence of $L\{1\}$.

(ii) To find Laplace transform of the function $F(t) = t^n$, *n being any real number greater than – 1.* **(Meerut 91)**

Sol. By definition, $L\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$.

$$\therefore \quad L\{t^n\} = \int_0^\infty e^{-st} t^n\,dt = \int_0^\infty e^{-st} t^{(n+1)-1}\,dt. \qquad \text{... (i)}$$

From the properties of 'Gamma function', we know that

$$\int_0^\infty e^{-ax} x^{m-1}\,dx = \frac{\Gamma(m)}{a^m}, \text{ if } a > 0 \text{ and } m > 0. \qquad \text{... (ii)}$$

Replacing a by s, m by $(n + 1)$ and x by t in *(ii)*, we have

$$\int_0^\infty e^{-st} t^{(n+1)-1}\,dt = \frac{\Gamma(n+1)}{s^{n+1}}, \text{ if } s > 0 \text{ and } n + 1 > 0$$

$$\therefore \quad (i) \Rightarrow L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}, s > 0 \text{ and } n > -1.$$

So the condition $s > 0$ is necessary for the convergence of the integral *(i)*.

(iii) To find Laplace transform of the function $F(t) = t^n$, *n being a positive integer.* **(Osmania 2004, Purvanchal 94, Andhra 90, Meerut 91)**

Sol. By definition, $L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$.

$\therefore \quad L\{t^n\} = \int_0^\infty e^{-st} t^n dt = \int_0^\infty e^{-st} t^{(n+1)-1} dt. \quad \ldots (i)$

From the properties of 'Gamma function', we know that

$$\int_0^\infty e^{-ax} x^{m-1} dx = \frac{\Gamma(m)}{a^m}, \text{ if } a > 0 \text{ and } m > 0. \quad \ldots (ii)$$

Replacing a by *s*, *m* by $(n + 1)$ and *x* by *t* in (*ii*), we have

$$\int_0^\infty e^{-st} t^{(n+1)-1} dt = \frac{\Gamma(n+1)}{s^{n+1}}, \text{ if } s > 0.$$

[$\because$ *n* is a positive integer $\Rightarrow (n + 1) > 0$]

$\therefore \quad (i) \Rightarrow L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$, if $s > 0 = \frac{n!}{s^{n+1}}$ if $s > 0$

[$\because n$ is a positive integer $\Rightarrow \Gamma(n + 1) = n!$]

Here the condition $s > 0$ is necessary for the convergence of the integral (*i*).

(*iv*). *Laplace transform of the function* $F(t) = e^{at}$.

(*Purvanchal 96, Andhra 90, Meerut 91*)

Sol. By definition, $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$.

$\therefore \quad L\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt, = \left[-\frac{e^{-(s-a)t}}{s-a} \right]_0^\infty$

or $\quad L\{e^{at}\} = 1/(s-a)$ provided $s > a$.

Since the integral involved in the above proof is convergent if $s > a$ and divergent if $s \le a$, we must take $s > a$ for the existence of $L\{e^{at}\}$.

(*v*) *Laplace of the function* $F(t) = \sin at$. **(*Kanpur 94,95, Meerut 99*)**

Sol. By definition, $L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$.

$\therefore \quad L\{\sin at\} = \int_0^\infty e^{-st} \sin at\, dt = \left[\frac{e^{-st}}{s^2 + a^2} (-s \sin at - a \cos at) \right]_0^\infty$

$$\left[\because \int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) \right]$$

$= a/(s^2 + a^2)$, provided $s > 0$.

Here the condition $s > 0$ is necessary for the convergence of the integral involved in the above proof.

(*vi*) *Laplace transform of the function* $F(t) = \cos at$. **(*Kanpur 95*)**

Sol. By definition, $L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$

Sol. By definition, $L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$

$$\therefore \text{ L}\{\cos at\} = \int_0^\infty e^{-st} \cos at\, dt = \left[\frac{e^{-st}}{s^2+a^2}(-s\cos at + a \sin at)\right]_0^\infty$$

$$\left[\because \int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b \sin bx)\right]$$

$= s/(s^2 + a^2)$, provided $s > 0$.

Here the condition $s > 0$ is necessary for the convergence of the integral involved in the above proof.

(*vii*) *Laplace transform of the function F (t) = sinh at.*

(*Meerut 91, Rohilkhand 88*)

Sol. $L\{\sinh at\} = \text{L}\left\{\frac{1}{2}(e^{at} - e^{-at})\right\} = \frac{1}{2}L\{e^{at}\} - \frac{1}{2}L\{e^{-at}\}$

$= \frac{1}{2}\frac{1}{s-a} - \frac{1}{2}\frac{1}{s+a}$, if $s > a$ and $s > -a$ *i.e.*, $|s| > a$

[using result of part (*iv*)]

$= \frac{1}{2}\frac{(s+a)-(s-a)}{(s-a)(s+a)} = \frac{a}{s^2-a^2}$, if $|s| > a$.

Ex.1. Find the Laplace transform of the function $F(t) = (e^{at} - 1)/a$.

(*Meerut 92, 94*)

Sol. L {F (*t*)}

$= \text{L}\{(1/a)(e^{at} - 1)\} = (1/a)\text{ L}\{e^{at} - 1\} = (1/a)[\text{L}\{e^{at}\} - \text{L}\{1\}]$

$= \frac{1}{a}\left(\frac{1}{s-a} - \frac{1}{s}\right) = \frac{1}{s(s-a)}$, if $s > a$ and $s > 0$.

Ex.2. *Find the Laplace transform of* $F(t) = (\sin t - \cos t)^2$.

Sol. $L\{F(t)\}$

$= \text{L}\{(\sin t - \cos t)^2\} = \text{L}\{\sin^2 t + \cos^2 t - 2\sin t \cos t\}$

$= \text{L}\{1 - \sin 2t\} = \text{L}\{1\} - \text{L}\{\sin 2t\} = \frac{1}{s} - \frac{2}{s^2+2^2}$, $s > 0$

$= (s^2 - 2s + 4)/s(s^2 + 4)$, $s > 0$.

First shifting (or first translation) theorem.

If $L\{F(t)\} = f(s)$, *then* $L\{e^{at} F(t)\} = f(s-a)$, *where a is any real or complex constant*

Or If $f(s)$ is the Laplace transform of F (t), then $f(s-a)$ is the Laplace transform of e^{at} F (t), where a is any real or complex number.

(Osmania 2004, Kanpur 95, Meerut 91, Purvanchal 92, 94)

Proof. By definition, we have

$$f(s) = \text{L}\{\text{F}(t)\} = \int_0^{\infty} e^{-st} F(t)\, dt. \quad \text{... (1)}$$

Replacing s by $(s-a)$ on both sides of (1), we get

$$f(s-a) = \int_0^{\infty} e^{-(s-a)} F(t)\, dt = \int_0^{\infty} e^{-st} \{e^{at} F(t)\} dt$$

$$= \text{L}\{e^{at} F(t)\}, \text{ by definition of Laplace transform.}$$

Ex.1. *Evaluate (i) L $\{e^{at} \cos bt\}$ (ii) L $\{e^{-at} \sin bt\}$.*

Sol. (*i*) Since L $\{\cos bt\} = s/(s^2 + b^2) = f(s)$, (say)

hence by first shifting theorm, we have

$$\text{L}\{e^{at} \cos bt\} = f(s-a) = \frac{(s-a)}{(s-a)^2 + b^2}, \text{ using (1).}$$

(*ii*) Since L $\{\sin bt\} = b/(s^2 + b^2) = f(s)$, (say)

hence by first shifting theorem, we have

$$\text{L}\{e^{-at} \sin bt\} = f(s+a) = \frac{b}{(s+b)^2 + b^2}, \text{ using (2)}$$

Ex.2. *Evaluate (i) L $\{e^{4t} \cosh 5t\}$.*

(*ii*) L $\{e^{-4t} \cosh 2t\}$ (*iii*) L $\{e^{at} \sinh bt\}$.

Second shifting (or second translation) theorem.

$$\textit{If } \text{L}\{\text{F}(t)\} = f(s) \textit{ and } \text{G}(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$$

then $\text{L}\{\text{G}(t)\} = e^{-as} f(s)$.

Proof. By definition of Laplace transform, we have

$$\text{L}\{\text{G}(t)\} = \int_0^{\infty} e^{-st} G(t)\, dt = \int_0^{a} e^{-st} G(t)\, dt + \int_a^{\infty} e^{-st} G(t)\, dt, 0 < a < \infty$$

$$= \int_0^{a} e^{-st} \cdot 0\, dt + \int_a^{\infty} e^{-st} F(t-a)\, dt, \text{ putting values of G }(t)$$

$= 0 + \int_a^\infty e^{-st} F(t-a)\,dt = \int_0^\infty e^{-s(a+u)} F(u)\,du$

[Putting $t - a = u$ so that $dt = du$. Also note that when $t = a$, $u = 0$ and when $t = \infty$, $u = \infty$]

$= e^{-sa} \int_0^\infty e^{-su} F(u)\,du = e^{-sa} \int_0^\infty e^{-st} F(t)\,dt$

[using a property of definite integral]

$= e^{-sa}$ L {F (t)}, by definition of Laplace transform

$= e^{-sa} f(s)$, using given result L {F (t)} $= f(s)$

Ex.1. *Find L {G (t)}, where* $G(t) = \begin{cases} e^{t-a}, & t > a \\ 0, & t < a. \end{cases}$

Sol. By second shifting theorem, if L {F (t)} $= f(s)$

and $\quad G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a. \end{cases}$, then L {G (t)} $= e^{-as} f(s)$.

On comparision with the given value of $G(t)$, F $(t) = e^t$ and $a = a$.

Now, $f(s)$ = L {F (t)} = L $\{e^t\} = 1/(s-1)$, $s > 1$.

$\therefore$ By second shifting theorem L $\{G(t)\} = e^{-as} f(s) = \dfrac{e^{-as}}{s-1}$, $s > 0$,

Ex.2. *Find L {G (t)}, where*

$G(t) = \begin{cases} \sin(t - \pi/3), & t > \pi/3 \\ 0, & t < \pi/3. \end{cases}$ ***(Rohilkhand 97)***

Sol. By second shifting theorem, if L {F (t)} $= f(s)$.

and $G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$, then L {G (t)} $= e^{-as} f(s)$.

On comparison with the given value of $G(t)$, here F $(t) = \sin t$, $a = \pi/3$.

Now, $f(s)$ = L {F (t)} = L $\{\sin t\} = 1/(s^2 + 1)$

$\therefore$ by second shifting theorem, L {G (t)} $= e^{-as} f(s) = \dfrac{e^{-\pi s/3}}{s^2+1}$, $s > 0$

Change of scale property.

If L {F (t)} = f(s), then L {F (at)} = (1/a) f (s/a).

Proof. By definition, $\int_0^\infty e^{-st} F(t)\, dt = L\{F(t)\} = f(s)$

Now, L {F (at)} = $\int_0^\infty e^{-st} F(at)\, dt$

$$= \frac{1}{a}\int_0^\infty e^{-su/a} F(u)\, du, \quad \text{[Putting } at = u \text{ so that } a\, dt = du]$$

$$= \frac{1}{a}\int_0^\infty e^{-(s/a)t} F(t)\, dt = \frac{1}{a} f\left(\frac{s}{a}\right), \text{ using (1)}$$

Ex.1. *If L {F (t)} = $(s^2 - s + 1)/(2s + 1)^2 (s - 1)$ prove that L {F (2t)} = $(s^2 - 2s + 4)/4 (s + 1)^2 (s - 2)$.*

Sol. Given L {F (t)} = $\dfrac{s^2 - s + 1}{(2s+1)^2 (s-1)} = f(s)$, say

∴ by the change of scale property, we have

$$L\{F(2t)\} = \frac{1}{2} f\left(\frac{s}{2}\right) = \frac{1}{2}\,\frac{(s/2)^2 - (s/2) + 1}{[2\times(s/2)+1]^2[(s/2)-1]} = \frac{s^2 - 2s + 4}{4(s+1)^2 (s-2)}.$$

Ex.4. *Applying the change of scale property, find*

(*i*) L {sin 5*t*} (*ii*) L {cos 4*t*}

(*ii*) L {sinh 5*t*} (iv) L {cosh 5*t*}.

1.15. Laplace transforms of derivatives.

Theorem I. *Let F (t) be continuous for all t ≥ 0 and be of exponential order σ as t → ∞ and if F′ (t) is of class A, then Laplace transform of the derivative F′(t) exists when s > σ and* L {F′ (t)} = sL {F (t)} – F (0).

Proof. We divide the proof in two parts.

Case I Let F′ (t) be continuous for all $t \geq 0$. Then

$$L\{F'(t)\} = \int_0^\infty e^{-st} F'(t)\, dt$$

$$= \left[e^{-st} F(t)\right]_0^\infty - \int_0^\infty (-s) e^{-st} F(t)\, dt, \text{ integrating by parts}$$

$$= \lim_{t\to\infty} e^{-st} F(t) - F(0) + s\int_0^\infty e^{-st} F(t)\, dt$$

or $$L\{F'(t)\} = \lim_{t\to\infty} e^{-st} F(t) - F(0) + sL\{F(t)\}. \quad \ldots (1)$$

Since F (t) is of exponential order σ, we can find constant m (> 0)

such that $|F(t)| \leq me^{\sigma t}$ for all $t \geq 0$. ... (2)

$\therefore$ $|e^{-st} F(t)| = e^{-st} |F(t)| \leq e^{-st} m e^{\sigma t}$, using (2)

$\therefore$ $|e^{-st} F(t)| \leq me^{-(s-\sigma)t}$ ... (3)

Now, for $s > \sigma$, as $t \to \infty$, $me^{-(s-\sigma)t} \to 0$.

$\therefore$ From (3), $\lim_{t\to\infty} e^{-st} F(t) = 0$ for $s > \sigma$.

So when $s > \sigma$, from (1) we see that L {F′ (t)} exists and is given by

$$L\{F'(t)\} = s\,L\{F(t)\} - F(0). \quad \ldots (4)$$

Cast II. Let F′ (t) be piece-wise continuous. Then

$$L\{F'(t)\} = \int_0^\infty e^{-st} F'(t) dt. \quad \ldots (5)$$

We now break up R.H.S. as the sum of integrals in different ranges from 0 to ∞ such that F′ (t) is continuous in each of these sub-intervals. We can now apply the procedure of case I in each such interval and finally obtain as in above case I,

$$L\{F'(t)\} = s\,L\{F(t)\} - F(0).$$

Theorem II. *Let F (t) and F′ (t) be continuous functions for all t ≥ 0 and F″ be of exponential order σ as t → ∞ and if F″ (t) is of class A, then Laplace transform of F″ (t) exists when s > σ, and is given by*

$$L\{F''(t)\} = s^2 L\{F(t)\} - s\,F(0) - F'(0).$$

Theorem III. *Let F (t), F′ (t) and be F″ (t) be continuous for all t ≥ 0 and be of exponential order σ as t → ∞ and if F‴ (t) is of class A then Laplace transform of F‴ (t) exists when s > σ and is given by*

$$L\{F'''(t)\} = s^3 L\{F(t)\} - s^2 F(0) - s F'(0) - F''(0).$$

Theorem IV. Laplace transform of the n^{th} order derivative. General case. *Let $F(t)$ and its derivative $F'(t), F''(t) \ldots\ldots, F^{(n-1)}(t)$ be continuous for all $t \geq 0$ and be of exponential order σ as $t \to \infty$ and if $F^{(n)}(t)$ is of class A, then Laplace transform of $F^{(n)}(t)$ exists when $s > \sigma$, and is given by*

$$L\{F^n(t)\} = s^n L\{F(t)\} - s^{n-1}F(0) - s^{n-2}F'(0) - \ldots\ldots - F^{(n-1)}(0),$$

where $F^{(n)}(t) = \dfrac{d^n F(t)}{dt^n}$ .etc.

Ex.1. *If* $L\left\{2\sqrt{\left(\frac{t}{\pi}\right)}\right\} = \frac{1}{s^{3/2}}$, *show* $\frac{1}{s^{1/2}} = L\left\{\frac{1}{\sqrt{(\pi t)}}\right\}$. ***(Mee***

Sol. Let $F(t) = 2\sqrt{(t/\pi)}$ .Then $F(0) = 2\sqrt{(0/\pi)} = 0$.

Again, $F'(t) = \dfrac{d}{dt}\left[\dfrac{2}{\sqrt{\pi}}t^{1/2}\right] = \dfrac{2}{\sqrt{\pi}}\dfrac{1}{2}t^{-1/2} = \dfrac{1}{\sqrt{(\pi t)}}$.

Now, we know that (refer theorem I, Art. 1.15)

$$L\{F'(t)\} = s\,L\{F(t)\} - F(0).$$

Substituting values of $F'(t)$, $F(t)$ and $F(0)$ in (3), we get

$$L\left\{\frac{1}{\sqrt{(\pi t)}}\right\} = s\,L\left\{2\sqrt{\left(\frac{t}{\pi}\right)}\right\} - 0 = s\cdot\frac{1}{s^{3/2}} = \frac{1}{s^{1/2}}.$$

$[\because$ given that $L\{2\sqrt{t/\pi}\} = 1/s^{3/2}]$

Ex.2. *If* $L\{\sin\sqrt{t}\} = \dfrac{\sqrt{\pi}}{2s^{3/2}}e^{-1/4s}$, *show* $L\left\{\dfrac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\left(\dfrac{\pi}{s}\right)}e^{-1/4s}$

Sol. Let $F(t) = \sin\sqrt{t}$ so that $F(0) = 0$. ... (1)

Again, $F'(t) = \dfrac{d}{dt}\sin t^{1/2} = \cos t^{1/2}\cdot\dfrac{1}{2}t^{-1/2} = \dfrac{\cos\sqrt{t}}{2\sqrt{t}}$ ... (2)

Now, we know that (refer theorem I, Art. 1.15)

$$L\{F'(t)\} = s\,L\{F(t)\} - F(0). \quad \text{... (3)}$$

Substituting values of F′ (*t*), F (*t*) and F (0) in (3), we get

$$L\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = s\,L\{\sin\sqrt{t}\} - 0 = s \cdot \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$$

$[\because$ given that $L\{\sin\sqrt{t}\} = (1/2s^{3/2})\sqrt{\pi}e^{-1/4s}]$

or $\quad \dfrac{1}{2}\,L\left\{\dfrac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \dfrac{\sqrt{\pi}}{2s^{1/2}} e^{-1/4s}$ or $L\left\{\dfrac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\left(\dfrac{\pi}{s}\right)}\, e^{-1/4s}$.

Ex.4. *Find* $L\{f(t)\}$ *if* $f''(t) + 3f'(t) + 2f(t) = 0, f(0) = 1, f'(0) = 2.$

[*S.V. University (A.P.) 1997*]

Sol. Given $\quad f''(t) + 3f'(t) + 2f(t) = 0$ ... (1)

with $\quad f(0) = 1$ and $f'(0) = 2$ ... (2)

Taking Laplace transform of both sides of (1), we get

$L\{f''(t)\} + 3L\{f'(t)\} + 2L\{f(t)\} = L(0) = 0$

or $s^2 L\{f(t)\} - s f(0) - f'(0) + 3[s L\{f(t)\} - f(0)] + 2L\{f(t)\} = 0$

(Using results of theorems I and II of Art 1.15)

or $s^2 L\{f(t)\} - s - 2 + 3[s L\{f(t)\} - 1] + 2L\{f(t)\} = 0$, using (2)

or $(s^2 + 3s + 2) L\{f(t)\} = s + 5$ or $L\{f(t)\} = (s+5)/(s^2 + 3s + 2)$.

1.18 Laplace transforms of integrals.

Theorem. If $L\{F(t)\} = f(s)$, then $L\left\{\int_0^t F(x)\,dx\right\} = \dfrac{f(s)}{s}$.

(Lucknow 97, Purvanchal 95, Kanpur 94, 95, Rohilkhand 90)

Sol. Let $G(t) = \int_0^t F(x)\,dx$. Then $G'(t) = F(t)$ and $G(0) = 0$ ... (1)

Now, we know that (refer theorem I, Art 1.15)

$L\{G'(t)\} = s\,L\{G(t)\} - G(0) = s\,L\{G(t)\}$, using (1).

Ex.1. *Prove that* $L\left[\int_0^t \sin 2u\,du\right] = \dfrac{2}{s(s^2+4)}$. ***(Bangalore 97***

Sol. Here $L\{\sin 2t\} = 2/(s^2 + 2^2) = f(s)$, say ... (1

$\therefore \quad L\left[\int_0^t \sin 2u\,du\right] = \dfrac{f(s)}{s} = \dfrac{2}{s(s^2+4)}$, by (1)

Ex.2. *Evaluate* $L\left[\int_0^t \dfrac{\sin x}{x}\,dx\right]$. ***(Kumaun 1997, Delhi 97, Meerut 92)***

Sol. Here $L\{\sin t\} = 1/(s^2+1) = f(s)$, say. (1)

$\therefore \quad L\left\{\dfrac{\sin t}{t}\right\} = \int_s^\infty f(s)\,ds = \int_s^\infty \dfrac{1}{s^2+1}\,ds = \left[\tan^{-1} s\right]_s^\infty$, using (1)

$= \tan^{-1}\infty - \tan^{-1} s = (\pi/2) - \tan^{-1} s = \cot^{-1} s$, as $\tan^{-1} x + \cot^{-1} x = \pi/2$

$\therefore \quad L\left[\int_0^t \dfrac{\sin x}{x}\,dx\right] = \dfrac{\cot^{-1} s}{s}$.

1.19. Initial-value theorem and final-value theorem.

Theorem I, Initial value theorem. Let F (t) be continuous for all $t \geq 0$ and be of exponential order as $t \to \infty$ and if $F'(t)$ is of class A, then

$$\lim_{t \to 0} F(t) = \lim_{s \to \infty} sL\{F(t)\}.$$

Proof. We know that $\quad L\{F'(t)\} = sL\{F(t)\} - F(0)$

or $$\int_0^\infty e^{-st} F'(t)\, dt = sL\{F(t)\} - F(0). \qquad \text{... (1)}$$

Taking limit as $s \to \infty$ in (1), we get

$$\int_0^\infty \lim_{s \to \infty} [e^{-st} F'(t)] dt = \lim_{s \to \infty} [sL\{F(t)\} - F(0)]. \qquad \text{... (2)}$$

Since $F'(t)$ is sectionally continuous and of exponential order, we have

$\int_0^\infty \lim_{s \to \infty} [e^{-st} F'(t)]\, dt = 0$. Then (2) reduces to

$$0 = \lim_{s \to \infty} sL\{F(t)\} - F(0). \qquad \text{... (3)}$$

or $\lim_{t \to 0} F(t) = \lim_{s \to \infty} sL\{F(t)\}$, since $F(0) = \lim_{t \to 0} F(t)$.

Theorem II. Final-Value Theorem. *Let F (t) be continuous for all $t \geq 0$ and be of exponential order and if $F'(t)$ is of class A, then*

$$\lim_{t \to \infty} F(t) = \lim_{s \to 0} sL\{F(t)\}.$$ ***(Osmania 2004)***

Proof. We know that $\quad L\{F'(t)\} = sL\{F(t)\} - F(0)$

or $$\int_0^\infty e^{-st} F'(t)\, dt = sL\{F(t)\} - F(0). \qquad \text{... (1)}$$

Taking limit as $s \to 0$ in (1), we get

$$\lim_{s \to 0} \int_0^\infty e^{-st} F'(t) dt = \lim_{s \to 0} [sL\{F(t)\} - F(0)]$$

or $\int_0^\infty F'(t) dt = \lim_{s \to 0} sL\{F(t)\} - F(0)$ or $[F(t)]_0^\infty = \lim_{s \to 0} sL\{F(t)\} - F(0)$

or $\lim_{t \to \infty} F(t) - F(0) = \lim_{s \to 0} sL\{F(t)\} - F(0)$ or $\lim_{t \to \infty} F(t) = \lim_{s \to 0} sL\{F(t)\}$.