1.1 Introduction: Infinite series designed to represent general periodic functions in terms of simple ones (e.g Sines and cosines of angle and its multiples). Fourier series is more general than Taylor Series because many discontinuous periodic functions of Practical interest can developed in Fourier Series. Fourier series is possible for continuous functions, periodic functions and functions discontinuous in their values and derivatives. Fourier integrals and fourier Transforms extend the ideas and techniques of fourier series to non-periodic functions and have basic applications to PDEs.

The Fourier series allows us to model any arbitrary periodic signal or function $f(x)$ in the form

$$f(x) = \frac{a_0}{2} + (a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots) + (b_1 \sin x + b_2 \sin 2x + a_3 \sin 3x + \cdots)$$

$$= \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

under some condition the interval $[c, c+2l]$ as given below called Dirichlet's conditions

(i) $f(x)$ is periodic with a period $2l$.
(ii) $f(x)$ and its integrals are finite and single valued in $[c, c+2l]$
(iii) $f(x)$ is piecewise continuous in the interval $[c, c+2l]$
(iv) $f(x)$ has a finite no of maxima and minima in $[c, c+2l]$.

PERIODIC FUNCTION

A function $f(x)$ is said to be periodic if there exists positive number T such that $f(x+T) = f(x)$ $\forall\, x \in R$.
Here T is the smallest positive real number such that $f(x+T) = f(x)$ $\forall\, x \in R$ and is called the fundamental period of $f(x)$.
If $f(x) = f(x+T) = f(x+2T) = \ldots$ then T is called the period of function $f(x)$.

We know that $\sin x$, $\cos x$, $\sec x$, $\operatorname{cosec} x$ are periodic functions with period 2π whereas $\tan x$ and $\cot x$ are period with a period π. The functions $\sin nx$ and $\cos nx$ are periodic with period $\frac{2\pi}{n}$. While fundamental period of $\tan nx$ is $\frac{\pi}{n}$.

Example:
1. $\sin(5x+3)$

Solution: $f(x) = \sin(5x+3)$
$\sin 5x$ is a periodic function with a period $\frac{2\pi}{5}$

$\therefore$ $f(x)$ is a periodic ~~function~~ with a period $\frac{2\pi}{5}$.

ii. $|\cos x|$

$$f(x) = |\cos x| = \sqrt{\cos x^2} = \sqrt{\frac{1+\cos 2x}{2}}$$

$$\because |x| = \sqrt{x^2}$$

Now $\cos 2x$ is a periodic function with a period $\frac{2\pi}{2} = \pi$.

$\therefore$ $f(x)$ is periodic with a period π.

iii. K, Constant.

$$f(x) = f(x+T) = K = f(x) \quad \forall x \in R.$$

$\therefore$ $f(x)$ is periodic function, but fundam period of $f(x)$ i.e. T can not be defined. Since it is the smallest positive real numb

iv. $\cos x + \frac{1}{3}\cos 2x + \frac{1}{2}\cos\frac{x}{3}$.

$$f(x) = \cos x + \frac{1}{3}\cos 2x + \frac{1}{2}\cos\frac{x}{3}.$$

Periods of $\cos x$, $\cos 2x$ and $\cos\frac{x}{3}$ are 2π, π and 6π respectively.

$\therefore$ $f(x)$ is periodic with a perio which is lowest common multiple i.e. (l.c.m) of $(2\pi, \pi, 6\pi) = 6\pi$.

1.3 EULER'S FORMULA:

If a function $f(x)$ satisfies Dirichlet's conditions, it can be expanded into Fourier series series given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

where,

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx \\ \text{and} \quad b_n &= \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx \end{aligned}\right\} \longrightarrow (A)$$

Proof: Let $f(x)$ be represented in the interval $[c,$ by the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \text{ —}$$

To find a_0:

Integrating both sides of (1) with respect to x with the limits c to $c+2\pi$.

$$\int_c^{c+2\pi} f(x)\,dx = \frac{a_0}{2}\int_c^{c+2\pi} dx + \sum_{n=1}^{\infty}\left[a_n \int_c^{c+2\pi} \cos nx\,dx + b_n \int_c^{c+2\pi} \sin nx\right.$$

$$= \frac{a_0}{2}(c+2\pi-c) + \sum_{n=1}^{\infty}\left[a_n \left(\frac{\sin nx}{n}\right)_c^{c+2\pi} + b_n\left(-\frac{\cos n}{n}\right.\right.$$

$$= \frac{a_0}{2} 2\pi + \sum_{n=1}^{\infty}(a_n \cdot 0 + b_n \cdot 0)$$

$= a_0 \pi.$

Hence $\boxed{a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x)\, dx}$ $\longrightarrow (2)$

To evaluate a_n:

Multiplying both sides of (1) by $\cos mx$ and integrat with respect to x within the limits c to $c+2\pi$.

$$\int_c^{c+2\pi} f(x) \cos nx\, dx = \frac{a_0}{2} \int_c^{c+2\pi} \cos mx\, dx + \sum_{n=1}^{\infty} a_0 \left[\int_c^{c+2\pi} \cos nx \cos mx\, dx \right.$$

$$+ \sum_{n=1}^{\infty} b_n \left[\int_c^{c+2\pi} \sin nx \cos mx\, dx \right]$$

$$\cancel{= 0 + a_n \int_c^{c+2\pi} \cos^2 x}$$

The first and third integrals on the right-hand side are always zero, but second integral is equal to π when $m=n$; otherwise it also vanishes when $m \neq n$.

$$\int_c^{c+2\pi} f(x) \cos nx\, dx = \frac{a_n}{2} \int_c^{c+2\pi} (1 + \cos 2nx)\, dx$$

$$= \frac{a_n}{2} \left[x + \frac{\sin 2nx}{2n} \right]_c^{c+2\pi}$$

$$= \frac{a_n}{2} \left[c + 2\pi - c \right] = a_n \pi.$$

$$\therefore a_n = \frac{1}{\pi} \int_c^{c+2\pi} f(x) \cos nx\, dx. \longrightarrow (3).$$

To find b_n;

Multiplying both sides of (1) by $\sin mx$ and integrating with respect to x within the limits c to $c+2\pi$.

$$\int_c^{c+2\pi} f(x)\sin mx\,dx = \frac{a_0}{2}\int_c^{c+2\pi}\sin mx\,dx + \sum_{n=1}^{\infty}\left[a_n\int_c^{c+2\pi}\cos nx\sin mx\,dx + b_n\int_c^{c+2\pi}\sin nx\sin mx\,dx\right]$$

The first two integrals on the right hand side are always zero. but the third integral is equal to π when $m=n$; otherwise vanish.

$$\int_c^{c+2\pi} f(x)\sin nx\,dx = \frac{a_0}{2}(0) + a_n(0) + \frac{1}{2}b_n\int_c^{c+2\pi} 2\sin^2 nx\,dx$$

$$= \frac{1}{2}b_n\int_c^{c+2\pi}(1-\cos 2nx)\,dx$$

$$= \frac{1}{2}b_n\left(x - \frac{\sin 2nx}{2n}\right)_c^{c+2\pi}$$

$$= \frac{1}{2}b_n(2\pi) = \pi\cdot b_n.$$

$$\therefore\ b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx \longrightarrow (4)$$

The results (2), (3) and (4) are called Euler's formula.

Some useful results in computation of the Fourier Seri[es]

If m, n are non-zero integers, then

(i) $\int_{c}^{c+2\pi} \sin nx \, dx = 0$

(ii) $\int_{c}^{c+2\pi} \cos nx \, dx = 0, \quad n \neq 0.$

(iii) $\int_{c}^{c+2\pi} \sin mx \sin nx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$

(iv) $\int_{c}^{c+2\pi} \cos mx \cos nx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$

(v) $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$

(vi) $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$

(V) $\sin n\pi = 0$

(VI) $\cos n\pi = (-1)^n$

(vii) Integration by parts when first function vanishes after a finite number of differentiation[s]

If u and v are function of x

$\int u.v \, dx = uv_1 - u^{(1)} v_2 + u^{(2)} v_3 - u^{(3)} v_4 + \cdots$

Here $u^{(n)}$ is derivatives of $u^{(n-1)}$ and v_n is inte[gral] of v_{n-1}

.4 Dirichlet's Condition

Following are the assumption for the expansion in a Fourier's series.

(i) The given function $f(x)$ is assumed to be defined and single valued in the given range $(-\pi, \pi)$

(ii) $f(x)$ is periodic outside $(-\pi, \pi)$ with the period 2π

(iii) We assumed that the series $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ is uniformly convergent, so that the term by term integration of the series is possible.

Then the series converges to

(a) $f(x)$ if x is a point of continuity

(b) $\frac{f(x+0) + f(x-0)}{2}$, if x is a point of discontinuity.

Also the series converges to $\frac{f(-\pi+0) + f(\pi-0)}{2}$ at $x = \pm\pi$ when these limits exist.

Here $f(x+0)$ is the right hand limit of $f(x)$ at x and represents $\lim_{h\to 0} f(x+h)$. Similarly $f(x-0)$ is the left hand limit of $f(x)$ at x and represents $\lim_{h\to 0} f(x-h)$.

~~1.4, Dirichlet's condition~~

1.5 CONDITIONS FOR FOURIER EXPANSION.

Dirichlet developed the Dirichlet criteria, which govern whether certain functions have valid Fourier expansions.

A function $f(x)$ has a valid Fourier Series expansion of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ where a_0, a_n, b_n are constants, provided

(i) $f(x)$ is well defined, periodic, single-valued and finite.

(ii) $f(x)$ has a finite number of finite discontinuiti in any one period.

(iii) $f(x)$ has a finite number of extrema (maxima and minima) within that period.

Working Rule for finding Fourier Series of $f(x)$ in th interval $(c, c+2\pi)$

Step 1: Fourier series $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Step 2: Compute a_0 and a_n, b_n using the formulae

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\, dx$$

and $$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \cos nx\, dx, \quad n = 1, 2, 3 - - -$$

$$b_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\sin nx\,dx \quad , n=1,2,3 \ldots$$

Step 3: Substitute a_0, a_n, b_n in $f(x)$ in step 1 to get the required Fourier series.

Example 1. Express $f(x) = \pi - x$ as Fourier series in the interval $0 < x < 2\pi$.

Solution:

Let $f(x) = \pi - x$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

Here, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$= \frac{1}{\pi}\int_0^{2\pi} (\pi - x)\,dx = \frac{1}{\pi}\left[\pi x - \frac{x^2}{2}\right] = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (\pi - x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{(\pi - x)}{n}\sin nx\right]_0^{2\pi} + \frac{1}{\pi}\int_0^{2\pi} \frac{\sin nx}{n}\,dx$$

$$= \frac{1}{\pi}\left[\frac{(\pi - x)}{n}\sin nx\right]_0^{2\pi} + \frac{1}{\pi}\left[-\frac{\cos nx}{n^2}\right]_0^{2\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\, \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (\pi - x)\, \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\Big]\, dx$$

$$= \frac{1}{\pi}\left[-\frac{(\pi - x)}{n}\cos nx - \frac{\sin nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi}{n}\cos 2n\pi - 0\right) - \left(-\frac{\pi}{n} - 0\right)\right]$$

$$= \frac{1}{\pi}\left(\frac{\pi}{n} + \frac{\pi}{n}\right) = \frac{2}{n}.$$

Substituting the values a_0, a_n, b_n in equation (1) we get

$$f(x) = \pi - x = \sum_{n=1}^{\infty} b_n \sin nx.$$

$$= \sum_{n=1}^{\infty} \frac{2}{n} \sin nx$$

$$= 2\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right]$$

Example 2. Find the Fourier series of the function $f(x) = x - \pi$ in the interval. $-\pi < x < \pi$.

Solution:

Here $f(x) = x - \pi$

Suppose $f(x) = x - \pi$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (x-\pi)\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\, dx - \pi \int_{\pi}^{\pi} dx\right]$$

$$= \frac{1}{\pi}\left(0 - \pi \cdot 2 \int_0^{\pi} dx\right)$$

$$= \frac{1}{\pi}(-2\pi)\left[x\right]_0^{\pi}$$

$$= -2\pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x-\pi) \cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x \cos nx\, dx - \pi \int_{-\pi}^{\pi} \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[0 - 2\pi \int_0^{\pi} \cos nx\, dx\right] \quad [\because x \cos nx - \text{odd funct}$$

$$= -\frac{2\pi}{\pi} \int_0^{\pi} \cos nx\, dx = -2\left(\frac{\sin nx}{n}\right)_0^{\pi}$$

$$= \cancel{-\frac{2}{n}(\sin n\pi} = -\frac{2}{n}(\sin n\pi - \sin 0) = -\frac{2}{n}(0-0)$$

$$= 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ \text{Sin}\,nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x-\pi)\ \text{Sin}\,nx\,dx.$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\,\text{Sin}\,nx\,dx - \pi\int_{-\pi}^{\pi}\text{Sin}\,nx\,dx\right]$$

$$= \frac{1}{\pi}\left[2\int_{0}^{\pi} x\,\text{Sin}\,nx\,dx - 0\right]$$ [∵ $x\,\text{Sin}\,nx$ – even f... $\text{Sin}\,nx$ – odd fu...

$$= \frac{2}{\pi}\int_{0}^{\pi} x\,\text{Sin}\,nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\text{Cos}\,nx}{n}\right) - \left(\frac{-\text{Sin}\,nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi\,\text{Cos}\,n\pi}{n} + 0 - (0+0)\right]$$ [∵ $\text{Sin}\,n\pi =$ ...

$$= \left(-\frac{2}{n}\right)\text{Cos}\,n\pi = \left(-\frac{2}{n}\right)(-1)^n$$

$$= \frac{2}{n}(-1)^{n+1}, \quad n = 1, 2, 3 \ldots$$

Substituting the values a_0, a_n, b_n in (1).

$$x - \pi = -\pi + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\,\text{Sin}\,nx.$$

$$= -\pi + 2\left[\text{Sin}\,x - \frac{1}{2}\text{Sin}\,2x + \frac{1}{3}\text{Sin}\,3x - \cdots\right.$$

Example. 3. Find a series of sines and cosines of multiples of x which will represent $(x+x^2)$ in the interval $-\pi<x<\pi$. Hence show that $\frac{\pi^2}{6} = 1+\frac{1}{2^2}+\frac{1}{3^2}+ \cdots$

Solution:

Here $f(x) = x+x^2$

Suppose $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left(\frac{2\pi^3}{3}\right) = \frac{2\pi^2}{3} \longrightarrow (1)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2\cos nx\,dx$$

[$\because$ $x\cos nx$ – odd function of x and $x^2\cos nx$ is an even function of x]

$$= \frac{2}{\pi}\left[x^2\left(\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[2\pi\left(\frac{\cos n\pi}{n^2}\right)\right] = \frac{4}{n^2}(-1)^n \quad \text{as } \sin n\pi = 0$$

$\longrightarrow (2)$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

[$\because x^2 \sin nx$ – is odd function of x]

$$= \frac{2}{\pi}\left[(x)\left(-\frac{\cos nx}{n}\right) - (1).\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi\left(-\frac{\cos n\pi}{n}\right) + 0\right]$$

$$= -\frac{2}{n}(-1)^n \longrightarrow (3)$$

Now putting values of a_0, a_n, b_n in (1), we get

$$f(x) = x + x^2$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left\{\frac{4}{n^2}(-1)^n \cos nx - \frac{2}{n}(-1)^n(\sin nx)\right\}.$$

$$= \frac{\pi^2}{3} - 4\left(\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \cdots\right)$$

$$+ 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right) \longrightarrow (4)$$

This is the required series.

Again putting $x = \pi$ and $x = -\pi$, successively in (4)

We get

$$\pi + \pi^2 = \frac{\pi^2}{3} + 4\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) \longrightarrow (5)$$

and $$-\pi + \pi^2 = \frac{\pi^2}{3} + 4\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) \quad \cdots (6)$$

Adding (5) and (6)

$$2\pi^2 = \frac{2\pi^2}{3} + 8\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)$$

$$\Rightarrow \frac{\pi^2}{6} = 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$$

Example 4. Find a Fourier series to represent $f(x)=x^2$ in the interval $(0, 2\pi)$

Solution:

Suppose $f(x) = x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right) \rightarrow (1)$

Here,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx$$

$$= \frac{1}{\pi}\left(\frac{x^3}{3}\right)_0^{2\pi} = \frac{1}{3\pi}\left(8\pi^3 - 0\right) = \frac{8}{3}\pi^2.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(0 + \frac{4\pi\cos 2n\pi}{n^2} - 0\right) - \left(0+0-0\right)\right]$$

$$= \frac{4}{n^2} \qquad (\because \cos 2n\pi = 1)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{x^2\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(-\frac{4\pi^2 \cos 2n\pi}{n} + 0 + \frac{2\cos 2n\pi}{n^3}\right) - \left(0+0+\frac{2}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{4\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3}\right] = -\frac{4\pi}{n}.$$

Substituting the values of a_0, a_n, b_n in (

$$x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx - \sum_{n=1}^{\infty} \frac{4\pi}{n} \sin nx$$

$$= \frac{4}{3}\pi^2 + 4\left(\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \cdots\right)$$

$$- 4\pi\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right)$$

Example 5. Obtain an expression in a mixed series of sines and cosines of multiples of which is zero between $-\pi$ and 0 and is equal to e^x between zero and π and give its value at three limits.

Solution: Here $f(x)$ is defined as follows

$$f(x) = \begin{cases} 0, & \text{for } -\pi < x < 0 \\ e^x, & \text{for } 0 < x < \pi. \end{cases}$$

Suppose,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \longrightarrow (1)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0.\,dx + \int_{0}^{\pi} e^{x}\,dx\right] = \frac{1}{2\pi}\left[e^{x}\right]_{0}^{\pi} = \frac{1}{2\pi}(e^{\pi}-1)$$

$\longrightarrow (2)$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$$

$$= 0 + \frac{1}{\pi}\int_{0}^{\pi} e^{x}\cos nx\,dx = \frac{1}{\pi(1+n^2)}\left[e^{x}(1.\cos nx + n\sin nx)\right]_{0}^{\pi}$$

$$= \frac{1}{\pi(n^2+1)}\left(e^{\pi}\cos n\pi - 1\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx.$$

$$= \frac{1}{\pi}\int_{0}^{\pi} e^{x}\sin nx\,dx$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{x}(\sin nx - n\cos nx)\right]_{0}^{\pi}$$

$$= -\frac{n(e^{x}\cos nx - 1)}{\pi(1+n^2)}$$

Putting values in (1), We get

$$f(x) = \frac{(e^{\pi}-1)}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(e^{\pi}\cos n\pi - 1)}{1+n^2}(\cos nx - n\sin nx)$$

$$= \frac{(e^{\pi}-1)}{2\pi} - \frac{(e^{\pi}+1)}{2\pi}(\cos x - \sin x) + \frac{(e^{\pi}-1)}{5\pi}(\cos 2x - 2\sin 2x)$$

Example 6. Find the Fourier Series of $f(x) = x - x^2$, $-\pi < x < \pi$. Hence show that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}.$$

Solution:

Let $f(x) = x - x^2$

Suppose $f(x) = x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

Now,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\,dx - \int_{-\pi}^{\pi} x^2\,dx\right]$$

($\because$ x is odd function)

$$= \frac{1}{\pi}\left[0 - 2\int_0^{\pi} x^2\,dx\right]$$

$$= -\frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = -\frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\frac{\sin nx}{n} - (1-2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{-4\cos n\pi}{n^2} = \frac{-4(-1)^n}{n^2} \quad [\because \cos n\pi = (-1)^n]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\,\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(\frac{-\cos nx}{n}\right) - (1-2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos n\pi}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{-2\cos n\pi}{n} = \frac{-2(-1)^n}{n}$$

Substituting the values a_0, a_n, b_n in (1), we get

$$x - x^2 = -\frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{-4(-1)^n}{n^2}\cos nx + \frac{-2(-1)^n \sin nx}{n}\right)$$

$$= \frac{-\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4(-1)^{n+1}}{n^2}\cos nx + 2\frac{(-1)^{n+1}}{n}\sin nx\right]$$

$$= \frac{-\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right]$$

$$+ 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right] \quad \rightarrow (2)$$

Putting $x=0$ in ~~equation~~ (2), we get

$$0 = \frac{-\pi^2}{3} + 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right)$$

i.e $$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + - \cdots = \frac{\pi^2}{12}$$

Example 7. Obtain the Fourier series for the function

$f(x) = x\sin x, \quad 0 < x < 2\pi$

Solution:

Let $f(x) = x\sin x$.

Suppose $$x\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Here,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx$$

$$= \frac{1}{\pi}\Big[x(-\cos x) - 1(-\sin x)\Big]_0^{2\pi}$$

$$= \frac{1}{\pi}\Big[-x\cos x + \sin x\Big]_0^{2\pi}$$

$$= -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\sin x\cos nx)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{2\pi}\Bigg[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - \left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\Bigg]_0^{2\pi}$$

$$= -\frac{1}{n+1} + \frac{1}{n-1} = \frac{2}{n^2-1} \quad (n \neq 1)$$

If $n=1$,

$$a_1 = \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx = \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - \left(\frac{-\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}(-\pi) = \frac{-1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\,\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\,(2\sin x\,\sin nx)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(n-1)x - \cos(n+1)x\right]dx$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right\} - \left\{\frac{-\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$\therefore\ b_n = 0$ for $n \neq 1$

If $n=1$, then

$$b_1 = \frac{1}{2\pi}\int_0^{2\pi} x.\,2\sin^2 x\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\,(1-\cos 2x)\,dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - \left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi.2\pi - \frac{4\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right] = \pi.$$

Substetuting the values of a_0, a_n and b_n in(1)

~~x Sinx =~~

$$x\sin x = -1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx + \pi \sin x.$$

$$= \frac{K}{\pi}\left[(\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\frac{\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{K}{\pi}\left[\left\{\frac{2\pi^2}{n} + 0 - \frac{2}{n^3}\right\} - \left\{0 + 0 - \frac{2}{n^3}\right\}\right]$$

$$= \frac{2K\pi}{n}$$

Substetuting the values of a_0, a_n, b_n in(1)

$$f(x) = -\frac{K\pi^2}{3} - 4K\sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx + 2K\pi \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

6 EVEN AND ODD FUNCTIONS OF x (Sine Series and cosine Series)

A function $f(x)$ is said to be an even function of x if $f(-x) = f(x)$

and then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

A function $f(x)$ is said to be an odd function of x if $f(-x) = -f(x)$ and then $\int_{-a}^{a} f(x)\,dx = 0.$

Thus we know that a function $f(x)$ defined in $-\pi < x < \pi$ can be represented by the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

Case I: When $f(x)$ is an even function.

We know that $\cos nx$ is an even function while $\sin nx$ is an odd function. Therefore $f(x)\cos nx$ is also an even function and $f(x)\sin nx$ is an odd function

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx = 0.$

$\therefore$ $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$ becomes

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx}$$

Case 2: When $f(x)$ is an odd function.

In this case $f(x)\cos nx$ is an odd function and $f(x)\sin nx$ is an even function.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$

$\therefore$ $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$ becomes

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n\sin nx}$$

where $b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$.

Example 1. Find the Fourier Series for the function $f(x) = x$, in $-\pi < x < \pi$ and deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$.

Solution: Since x is an odd function of x.

$$f(-x) = -x = -f(x)$$

$$\therefore \quad f(x) = x = \sum_{n=1}^{\infty} b_n \sin nx$$

Where,

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\cdot\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{4}\cos n\pi + 0\right] \qquad \left[\because \sin n\pi = 0\right]$$

$$= \frac{-2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$$

$\therefore$ The series is

$$x = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx$$

$$= 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right] \rightarrow (1)$$

Putting $x = \frac{\pi}{2}$ in (1), we get

$$\frac{\pi}{2} = 2\left[\sin\frac{\pi}{2} - \frac{1}{2}\sin\frac{2\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} - \cdots\right]$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{2}\cdot 0 + \frac{1}{3} + \frac{1}{4}\cdot 0 + \frac{1}{5} - \cdots$$

$$= 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$

Hence.

Example 2. Obtain Fourier's Series for the expansion of $f(x) = x\sin x$ in the interval $(-\pi, \pi)$, Hence deduce

$$\frac{\pi-2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + - - - .$$

Solution: Here $f(x) = x\sin x$

Clearly $x\sin x$ is an even function of x.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \longrightarrow (1)$$

where $a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin x\,dx$

$$= \frac{1}{\pi}\Big[x(-\cos x) - (1)(-\sin x)\Big]_0^{\pi} = \frac{1}{\pi}\cdot\pi = 1$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\big\{\sin(n+1)x + \sin(1-n)x\big\}\,dx$$

$$= \frac{1}{\pi}\Big[x\Big\{-\frac{\cos(n+1)x}{n+1} - \frac{\cos(1-n)x}{1-n}\Big\}$$

$$- (1)\cdot\Big\{-\frac{\sin(n+1)x}{(n+1)^2} - \frac{\sin(1-n)x}{(1-n)^2}\Big\}\Big]_0^{\pi}$$

$$= \cancel{\frac{1}{n+1}\cos n\pi +} \frac{\cos n\pi}{1-n} \cancel{\frac{2\cos n\pi}{(1-n)^2}\ (\text{if } n \neq 1)}$$

$$= \frac{1}{\pi}\Big[-\pi\Big(\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(1-n)\pi}{1-n}\Big)\Big]$$

~~Hence~~

$$= -\left[\frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1}\right] \quad \left[\because \cos(-x) = \cos x\right]$$

$$= -\left[\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}(-1)^2}{n-1}\right]$$

$$= -\left[\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n+1}}{n-1}\right]$$

$$= (-1)(-1)^{n+1}\left[\frac{n-1-n-1}{n^2-1}\right] = \frac{2(-1)^{n+1}}{n^2-1} \quad n \neq 1$$

If $n=1$, then

$$a_n \text{ (i.e } a_1) = \frac{2}{\pi}\int_0^\pi x \sin x \cos x\,dx$$

$$= \frac{1}{\pi}\int_0^\pi x \sin 2x\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - (1)\cdot\left(-\frac{\sin 2x}{4}\right)\right]_0^\pi$$

$$= \frac{1}{\pi}\left(-\frac{\pi\cos 2\pi}{2}\right) = -\frac{1}{2}$$

Hence (1) gives

$$\cancel{x\sin x = 1 + \left[-\frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{\cos n\pi}{1-n^2}\cos nx\right]}$$

$$x\sin x = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{2(-1)^{n+1}}{n^2-1}\cos nx.$$

$$= 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{2(-1)^{n+1}}{(n-1)(n+1)}\cos nx.$$

$$= 1 - \frac{1}{2}\cos x + \left(\frac{-2}{1\cdot 3}\cos 2x + \frac{2}{2\cdot 4}\cos 3x - \frac{2}{3\cdot 5}\cos 4x + \cdots\right) \rightarrow (2)$$

Putting $x = \frac{\pi}{2}$ in (2), we get

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1 - 0 - \frac{2}{1.3}(-1) + \frac{2}{2.4}(0) + \frac{(-2)}{3.5}(1) + \cdots$$

$$\Rightarrow \frac{\pi}{2} = 1 + \frac{2}{1.3} - \frac{2}{3.5} + \frac{2}{5.7} - \cdots$$

$$\Rightarrow \frac{\pi}{2} - 1 = \frac{2}{1.3} - \frac{2}{3.5} + \frac{2}{5.7} - \cdots$$

$$\Rightarrow \frac{\pi - 2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots$$

Example 3. Obtain Fourier's series for the expansion of $f(x) = x^2$ in $(-\pi, \pi)$, and hence deduce that

(i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - \cdots = \frac{\pi^2}{12}$

(ii) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$

Solution:

Since $f(-x) = (-x)^2 = f(x)$

$\therefore$ $f(x)$ is an even function.

Hence $x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$. $\longrightarrow (1)$

Where —

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{2\pi^2}{3} \longrightarrow (2)$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right)-(2x)\left(-\frac{\cos nx}{n^2}\right)+(2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0+2\pi\frac{\cos n\pi}{n^2}+2\cdot 0\right]$$

$$= \frac{2}{\pi}\,\frac{2\cos n\pi}{n^2} = \frac{4}{n^2}(-1)^n. \longrightarrow (3)$$

ow from equation (1)

$$x^2 = \frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx.$$

$$= \frac{\pi^2}{3}-4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx$$

$$= \frac{\pi^2}{3}-4\left[\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\frac{\cos 4x}{4^2}+\cdots\right] \longrightarrow (4)$$

utting $x=0$ in (4)

$$0 = \frac{\pi^2}{3}-4\left(1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\right)$$

$$\Rightarrow \frac{\pi^2}{12} = 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots \longrightarrow (5)$$

Putting $x=\pi$ in (4)

$$\pi^2 = \frac{\pi^2}{3}-4\left(\cos\pi-\frac{\cos 2\pi}{2^2}+\frac{\cos 3\pi}{3^2}-\frac{\cos 4\pi}{4^2}+\cdots\right)$$

$$\Rightarrow \cancel{\frac{\pi^2}{3}-\frac{\pi^2}{3}}$$

$$\Rightarrow \pi^2-\frac{\pi^2}{3} = -4\left(-1-\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{4^2}-\cdots\right)$$

$$\Rightarrow \frac{2\pi^2}{3} = 4\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)$$

$$\Rightarrow \frac{\pi^2}{6} = 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots \longrightarrow (6)$$

Adding ~~equation~~ (5) and (6), We get

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots=\frac{\pi^2}{8}.$$

Example: 4. Obtain the Fourier Series for the function $f(x)=|x|$ in $-\pi<x<\pi$, and deduce that

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}.$$

Solution: Since $f(-x)=|-x|=x=|x|=f(x)$

So, $f(x)$ is an even function.

$$\therefore f(x)=|x|=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0=\frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$$=\frac{2}{\pi}\int_0^{\pi}|x|\,dx=\frac{2}{\pi}\int_0^{\pi}x\,dx$$

$$=\frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi}=\pi.$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}|x|\cos nx\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx.$$

$$=\frac{2}{\pi}\left[x\frac{\sin nx}{n}-(1)\cdot\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\frac{\cos n\pi}{n^2}-\frac{1}{n^2}\right]=\frac{2}{n^2\pi}\left[(-1)^n-1\right]$$

$$\therefore a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ -\frac{4}{\pi n^2}, & \text{if } n \text{ is odd.} \end{cases}$$

Substituting a_0, a_n in (1), we get

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right) \longrightarrow (4)$$

Putting $x = 0$ in (4), we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Half Range Fourier Series

A series which ~~contains only~~ sine terms or cosine terms is called a half range Fourier sine or cosine series respectively. In this case the function f(x) is ~~generally~~ generaly defined in the interval $(0, \pi)$ which is half of the interval $(-\pi, \pi)$], therefore the name half range is assigned.

Now If f(x) is an even function, the Fourier cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$$

If $f(x)$ is an odd function of x the Fouri[illegible] sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Where,

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx .$$

Example 1. Expand $f(x) = \pi x - x^2$ in the interval $[0, \pi]$ and deduce that $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} - \dots = \frac{\pi^{[illegible]}}{3[illegible]}$

Solution:

Here $f(x) = \pi x - x^2 = \sum_{n=1}^{\infty} b_n \sin nx$

Where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x(\pi - x) \sin nx \, dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2}{n^3}(1 - \cos n\pi)\right]$$

$$= \frac{4}{\pi n^3}\left[1 - (-1)^n\right]$$

~~Hence~~ $\therefore$

$$b_n = \begin{cases} 0, & \text{when } n \text{ is even} \\ \frac{8}{\pi n^3}, & \text{,, } n \text{ is odd} \end{cases}$$

$$\text{Hence } (\pi x - x^2) = \sum_{n=1,3,5\ldots}^{\infty} \frac{8}{\pi n^3} \sin nx$$

$$= \frac{8}{\pi}\left(\sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots\right) \longrightarrow (1)$$

Which is required sine series.

Now, Putting $x = \frac{\pi}{2}$ in (1), we get

$$\frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{8}{\pi}\left(\sin\frac{\pi}{2} + \frac{1}{3^3}\sin\frac{3\pi}{2} + \frac{1}{5^3}\sin\frac{5\pi}{2} + \cdots\right)$$

$$\Rightarrow \frac{\pi^2}{4} = \frac{8}{\pi}\left[1 + \frac{1}{3^3}\sin\left(\pi + \frac{\pi}{2}\right) + \frac{1}{5^3}\sin\left(2\pi + \frac{\pi}{2}\right) + \cdots\right]$$

$$\Rightarrow \frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$$

Example 2. Obtain the half-range sine and cosine series for the function $f(x) = \frac{\pi x}{8}(\pi - x)$, $[0, \pi]$

Solution:

$$\text{Here } f(x) = \frac{\pi x}{8}(\pi - x) = \sum_{n=1} b_n \sin nx \longrightarrow (1)$$

$$\text{Where, } b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{\pi x}{8}(\pi - x) \sin nx \, dx$$

$$= \frac{2}{\pi}\left(\frac{\pi}{8}\right)\int_0^{\pi} (\pi x - x^2) \sin nx \, dx$$

$$= \frac{1}{4}\left[(\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\frac{2}{n^3}(1-\cos n\pi)\right]$$

$$= \frac{1}{2}\left[\frac{1-(-1)^n}{n^3}\right]$$

$$\therefore b_n = \begin{cases} 0, & \text{when } n \text{ is even} \\ \frac{1}{n^3}, & \text{when } n \text{ is odd.} \end{cases}$$

Substetuting b_n in (1)

$$\frac{\pi x}{8}(\pi - x) = \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^3} \sin nx$$

$$= \frac{1}{1^3}\sin x + \frac{1}{3^3}\sin 3x + \frac{1}{5^3}\sin 5x + \cdots$$

Again, The Fourier Cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Where, $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \frac{\pi x}{8}(\pi - x)\,dx$

$$= \frac{1}{4}\int_0^{\pi}(\pi x - x^2)\,dx$$

$$= \frac{1}{4}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{1}{4}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right]$$

$$= \frac{\pi^3}{24}$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^\pi \frac{\pi x}{8}(\pi - x)\cos nx\,dx$$

$$= \frac{1}{4}\int_0^\pi (\pi x - x^2)\cos nx\,dx$$

$$= \frac{1}{4}\left[(\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (\pi - 2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^\pi$$

$$= \frac{1}{4}\left[\left\{0 - \frac{\pi\cos n\pi}{n^2} + 0\right\} - \left\{0 + \frac{\pi}{n^2} + 0\right\}\right]$$

$$= \frac{1}{4}\left[\frac{-\pi\cos n\pi - \pi}{n^2}\right]$$

$$= \frac{-\pi}{4}\left[\frac{1 + \cos n\pi}{n^2}\right]$$

$$= \frac{-\pi}{4}\left[\frac{1 + (-1)^n}{n^2}\right]$$

$$\therefore a_n = \begin{cases} 0, & \text{when } n \text{ is odd} \\ -\frac{\pi^2}{2n^2}, & \text{when } n \text{ is even} \end{cases}$$

Putting the value a_0, a_n in (2), we get

$$f(x) = \frac{\pi^3}{48} - \frac{\pi}{2}\sum_{n=2,4,6\ldots}^{\infty} \frac{1}{n^2}\cos nx$$

$$= \frac{\pi^3}{48} - \frac{\pi}{2}\left(\frac{1}{2^2}\cos 2x + \frac{1}{4^2}\cos 4x + \frac{1}{6^2}\cos 6x + \cdots\right)$$

Example 3: Obtain the Fourier Cosine Series for $f(x) = x\sin x$, $0 < x < \pi$. and deduce that

$$\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots = \frac{\pi - 2}{4}$$

Solution:

Let $f(x) = x\sin x = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos nx \longrightarrow (1).$

where,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx$$

$$= \frac{2}{\pi}\Big[x(-\cos x) + (\sin x)\Big]_0^{\pi}$$

$$= \frac{2}{\pi}\Big[-\pi\cos\pi + \sin\pi\Big] = \frac{2}{\pi}(\pi) = 2.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x(2\sin x\cos nx)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\Big[\sin(n+1)x - \sin(n-1)x\Big]\,dx$$

$$= \frac{1}{\pi}\left\{x\left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right] - (1)\left[-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right]\right\}_0^{\pi}$$

$$= \frac{-1}{n+1}\cos(n+1)\pi + \frac{1}{n-1}\cos(n-1)\pi$$

$$= \frac{(-1)^{n-1}}{n-1} - \frac{(-1)^{n+1}}{n+1} = \frac{2(-1)^{n+1}}{n^2-1}$$

$a_2 = \frac{-2}{1.3}$, $a_3 = \frac{2}{2.4}$, $a_4 = \frac{-2}{3.5}$, ...

Now, $a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x \, dx$

$$= \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - \left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{2}\cos 2\pi\right] = -\frac{1}{2}$$

From (1), we have

$$x \sin x = 1 - \frac{1}{2}\cos x - \frac{2}{1.3}\cos 2x + \frac{2}{2.4}\cos 3x - \frac{2}{3.5}\cos 4x + \cdots$$

Now, Putting $x = \frac{\pi}{2}$ in (2),

$$\frac{\pi}{2} = 1 + \frac{2}{1.3} - \frac{2}{3.5} + \frac{2}{5.7} - \frac{2}{7.9} + \cdots$$

$$\Rightarrow 1 + \frac{2}{1.3} - \frac{2}{3.5} + \frac{2}{5.7} - \frac{2}{7.9} + \cdots = \frac{\pi}{2}$$

$$\Rightarrow \frac{2}{1.3} - \frac{2}{3.5} + \frac{2}{5.7} - \frac{2}{7.9} + \cdots = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}$$

$$\Rightarrow \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \cdots = \frac{\pi - 2}{4}$$

Example: 4. Find the half-range sine series of $f(x) = e^x$, $(0, \pi)$

Solution:

Let $f(x) = e^x = \sum_{n=1}^{\infty} b_n \sin nx$

Where, $b_n = \frac{2}{\pi}\int_0^{\pi} e^x \sin nx \, dx$.

$$b_n = \frac{2}{\pi}\left[\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\left[\frac{e^{\pi}}{1+n^2}(0 - n\cos n\pi) - \frac{1}{1+n^2}(0-n)\right\}\right.$$

$$= \frac{2}{\pi}\left[\frac{(-1)^{n+1} n e^{\pi}}{1+n^2} + \frac{n}{1+n^2}\right]$$

$$= \frac{2n}{\pi(1+n^2)}\left[1 + (-1)^{n+1} e^{\pi}\right]$$

Hence the half-range sine series for e^x is

$$f(x) = e^x = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n\left[1+(-1)^{n+1}e^{\pi}\right]}{n^2+1}\sin nx$$

$$= \frac{2}{\pi}\left[\frac{1+e^{\pi}}{1^2+1}\sin x + \frac{2(1-e^{\pi})}{2^2+1}\sin 2x + \frac{3(1+e^{\pi})}{3^2+1}\sin 3x + \cdots\right]$$